Proof that Bird's Linear Array Notation with 5 or more entries goes beyond Conway's Chained Arrow Notation

Conway's Chained Arrow Notation (invented by John Conway) operates according to the following rules:

When the chain consists of 3 entries, then

$a \rightarrow b \rightarrow c = a \uparrow\uparrow\uparrow\cdots\uparrow b$ (with c Knuth's up-arrows).

If the last entry in the chain is a 1, it can be removed:

$a \rightarrow b \rightarrow \ldots \rightarrow x \rightarrow 1 = a \rightarrow b \rightarrow \ldots \rightarrow x$.

If the penultimate entry in the chain is a 1, the last 2 entries can be removed:

$a \rightarrow b \rightarrow \ldots \rightarrow x \rightarrow 1 \rightarrow z = a \rightarrow b \rightarrow \ldots \rightarrow x$.

If there are just 2 entries in the chain, the remaining arrow becomes an exponent:

$a \rightarrow b = a \wedge b$ (since $a \rightarrow b \rightarrow 1 = a \wedge b$).

The last entry in the chain can be reduced by 1 by taking the penultimate entry and replacing it with a copy of the entire chain with its penultimate entry reduced by 1:

$a \rightarrow b \rightarrow \ldots \rightarrow x \rightarrow (y+1) \rightarrow (z+1)$

$= a \rightarrow b \rightarrow \ldots \rightarrow x \rightarrow (a \rightarrow b \rightarrow \ldots \rightarrow x \rightarrow y \rightarrow (z+1)) \rightarrow z$.

For example,

$a \rightarrow b \rightarrow \ldots \rightarrow x \rightarrow 2 \rightarrow (z+1)$

$= a \rightarrow b \rightarrow \ldots \rightarrow x \rightarrow (a \rightarrow b \rightarrow \ldots \rightarrow x \rightarrow 1 \rightarrow (z+1)) \rightarrow z$

$= a \rightarrow b \rightarrow \ldots \rightarrow x \rightarrow (a \rightarrow b \rightarrow \ldots \rightarrow x) \rightarrow z$

(1 nested bracket),

$a \rightarrow b \rightarrow \ldots \rightarrow x \rightarrow 3 \rightarrow (z+1)$

$= a \rightarrow \ldots \rightarrow x \rightarrow (a \rightarrow \ldots \rightarrow x \rightarrow 2 \rightarrow (z+1)) \rightarrow z$

$= a \rightarrow \ldots \rightarrow x \rightarrow (a \rightarrow \ldots \rightarrow x \rightarrow (a \rightarrow \ldots \rightarrow x \rightarrow 1 \rightarrow (z+1)) \rightarrow z) \rightarrow z$

$= a \rightarrow \ldots \rightarrow x \rightarrow (a \rightarrow \ldots \rightarrow x \rightarrow (a \rightarrow \ldots \rightarrow x) \rightarrow z) \rightarrow z$

(2 nested brackets).

In general,

$a \rightarrow b \rightarrow \ldots \rightarrow x \rightarrow (y+1) \rightarrow (z+1)$

$= a \rightarrow \ldots \rightarrow x \rightarrow (a \rightarrow \ldots \rightarrow x \rightarrow (\ldots\ldots (a \rightarrow \ldots \rightarrow x) \rightarrow z) \ldots\ldots) \rightarrow z$

(with y nested brackets).

The brackets can only be removed after the chain inside the brackets has been evaluated into a single number.

Almost the first thing that I did when learning of this new notation was that I conjectured the following results:

$\{a, b, 1, 2\} > a \rightarrow a \rightarrow (b-1) \rightarrow 2$ (for $a \geq 3$, $b \geq 2$),

$\{a, b, c, 2\} > a \rightarrow a \rightarrow (b-1) \rightarrow (c+1)$ (for $a \geq 3$, $b \geq 2$, $c \geq 1$),

$\{a, b, c, 3\} > a \rightarrow a \rightarrow a \rightarrow (b-1) \rightarrow (c+1)$ (for $a \geq 3$, $b \geq 2$, $c \geq 1$),

$\{a, b, c, d\} > a \rightarrow a \rightarrow a \rightarrow \ldots \rightarrow a \rightarrow (b-1) \rightarrow (c+1)$

(with d+2 entries in chain, for $a \geq 3$, $b \geq 2$, $c \geq 1$, $d \geq 2$).

Here, I will attempt to prove that

$$\{a, b, c, d\} > a \to a \to \ldots \to a \to (b-1) \to (c+1)$$

(with d+2 entries in chain, first d entries contain ‘a’),

for all $a \geq 3$, $b \geq 2$, $c \geq 1$, $d \geq 2$.

In order to do this, I first need to prove two Lemmas.

Lemma 1:

$$a \to b \to c \to \ldots \to y \to z \geq a,$$

for chains of any length, where $a, b, c, \ldots , y, z \geq 1$.

Let C represent the chain $a \to b \to c \to \ldots \to y \to z$, which is of any length, where all of the entries contain positive integers (1, 2, 3, ...).

When the chain is of length 1, $C = a$.

When the chain is of length 2, $C = a \to b = a$^$b \geq a$.

When the chain is of length 3 or longer, C can be reduced in length, one at a time (by operation of Conway’s Chained Arrow Notation), as follows,

$$\begin{aligned} C &= a \to b \to \ldots \to x \to y' \to (z-1) \\ &= a \to b \to \ldots \to x \to y'' \to (z-2) \\ &\quad \ldots\ldots \\ &= a \to b \to \ldots \to x \to y^* \to 1 \\ &= a \to b \to \ldots \to x \to y^* \\ &= a \to b \to \ldots \to x' \to (y^*-1) \\ &\quad \ldots\ldots \\ &= a \to b \to \ldots \to x^*, \end{aligned}$$

until they contain 3 entries, for example,

$$C = a \to b \to c^* = a \text{ ^^^}^{\cdots}\text{^ } b \qquad \text{(with } c^* \text{ Knuth's up-arrows).}$$

Using my ‘extended operator notation’ (where the number in curly brackets represents the number of up-arrows), C can be written as

$$C = a \{c^*\} b.$$

Since,

$$C = a \{c^*-1\} b' = a \{c^*-2\} b'' = \ldots = a \{1\} b^* = a\text{^}b^*,$$

for some positive integer b^*, it follows that $C \geq a$ for C of any length of 1 or greater, and so, Lemma 1 is proven.

Lemma 2:

$$a \to a \to \ldots \to a \to (y+1) \to z > (a \to a \to \ldots \to a \to y \to z) + 1$$

(with n+2 entries in chain on each side, first n entries contain ‘a’),

for all $a \geq 3$, $y \geq 1$, $z \geq 1$, $n \geq 1$.

This involves proof by induction.

When $a \ge 3$, $y \ge 1$, $z = 1$, $n = 1$,

$$
\begin{aligned}
a \to (y+1) &= a \wedge (y+1) \\
&= a \times a^y \\
&> 2a^y \\
&> a^y + 1 \qquad \text{(since } a^y \ge 3^1 > 1\text{)} \\
&> (a \to y) + 1,
\end{aligned}
$$

and so,

$$a \to (y+1) \to 1 > (a \to y \to 1) + 1,$$

this holds true.

Assuming that this holds true for $a \ge 3$, $y \ge 1$, $z = k$, $n = 1$,
when $a \ge 3$, $y \ge 1$, $z = k+1$, $n = 1$,

$$
\begin{aligned}
a \to (y+1) \to (k+1) &= a \{k+1\} (y+1) \qquad \text{(using my 'extended operator notation')} \\
&= a \{k\} (a \{k+1\} y) \qquad \text{(by definition)} \\
&= a \to (a \{k+1\} y) \to k \\
&= a \to (a \to y \to (k+1)) \to k \\
&> (a \to ((a \to y \to (k+1))-1) \to k) + 1 \\
&> (a \to ((a \to y \to (k+1))-2) \to k) + 2 \\
&\quad \ldots\ldots \\
&> (a \to 1 \to k) + (a \to y \to (k+1)) - 1 \\
&\qquad \text{(since } a \to y \to (k+1) \ge a > 1 \text{ by Lemma 1)} \\
&= a + (a \to y \to (k+1)) - 1 \\
&> (a \to y \to (k+1)) + 1,
\end{aligned}
$$

this holds true. So, Lemma 2 holds true for $a \ge 3$, $y \ge 1$, $z \ge 1$, $n = 1$.

Assuming that this holds true for $a \ge 3$, $y \ge 1$, $z \ge 1$, $n = k$,
when $a \ge 3$, $y \ge 1$, $z = 1$, $n = k+1$,

$$a \to a \to \ldots \to a \to a \to (y+1) = a \to a \to \ldots \to a \to N \to y$$

(with k+2 entries in chain on each side),

where $N = a \to a \to \ldots \to a \to (a-1) \to (y+1)$

(with k+2 entries in chain).

Since

$$a \to a \to \ldots \to a \to (y+1) \to z > (a \to a \to \ldots \to a \to y \to z) + 1$$

(with k+2 entries in chain on each side)

implies that

$$
\begin{aligned}
a \to a \to \ldots \to a \to y' \to z &> (a \to a \to \ldots \to a \to 1 \to z) + 1 \\
&> (a \to a \to \ldots \to a) + 1
\end{aligned}
$$

(for all $y' \ge 2$, where '$a \to a \to \ldots \to a$' denotes k entries of 'a'),

it follows that

$$
\begin{aligned}
N &> (a \to a \to \ldots \to a) + 1 \qquad \text{(with k entries in chain, since } a-1 \ge 2\text{)} \\
&\ge a+1 \qquad \text{(by Lemma 1),}
\end{aligned}
$$

and so,

$$
\begin{aligned}
a \to a \to \ldots \to a \to a \to (y+1) &> a \to a \to \ldots \to a \to (a+1) \to y \\
&> (a \to a \to \ldots \to a \to a \to y) + 1
\end{aligned}
$$

(with k+2 entries in chain on each side, since $N > a+1$).

This means that

$$a \to a \to \ldots \to a \to a \to (y+1) \to 1 > (a \to a \to \ldots \to a \to a \to y \to 1) + 1$$

(with k+3 entries in chain on each side),

and so Lemma 2 holds true for $a \ge 3$, $y \ge 1$, $z = 1$, $n = k+1$.

Assuming that this also holds true for $a \geq 3$, $y \geq 1$, $z = m$, $n = k+1$ (as well as $a \geq 3$, $y \geq 1$, $z \geq 1$, $n = k$), when $a \geq 3$, $y \geq 1$, $z = m+1$, $n = k+1$,

$$a \to a \to ... \to a \to (y+1) \to (m+1)$$

(with k+3 entries in chain and 'a → a → ... → a' represents k+1 entries of 'a')

$$= a \to a \to ... \to a \to (a \to a \to ... \to a \to y \to (m+1)) \to m$$
$$> (a \to a \to ... \to a \to ((a \to a \to ... \to a \to y \to (m+1))-1) \to m) + 1$$
$$> (a \to a \to ... \to a \to ((a \to a \to ... \to a \to y \to (m+1))-2) \to m) + 2$$

......

$$> (a \to a \to ... \to a \to 1 \to m) + (a \to a \to ... \to a \to y \to (m+1)) - 1$$

(since $a \to a \to ... \to a \to y \to (m+1) \geq a > 1$ by Lemma 1)

$$= (a \to a \to ... \to a \to y \to (m+1)) + (a \to a \to ... \to a) - 1$$
$$\geq (a \to a \to ... \to a \to y \to (m+1)) + a - 1$$

(since $a \to a \to ... \to a \geq a$ by Lemma 1)

$$> (a \to a \to ... \to a \to y \to (m+1)) + 1,$$

this holds true. So, this holds true for $a \geq 3$, $y \geq 1$, $z \geq 1$, $n = k+1$, which means that Lemma 2 holds true for $a \geq 3$, $y \geq 1$, $z \geq 1$, $n \geq 1$ and it is proven.

Corollary 1:

When $y' > y \geq 1$,

$$a \to a \to ... \to a \to y' \to z > a \to a \to ... \to a \to y \to z$$

(with n+2 entries in chain on each side, first n entries contain 'a'),

for all $a \geq 3$, $z \geq 1$, $n \geq 1$.

This is the result of Lemma 2 being applied repeatedly, since $y' > y'-1 > y'-2 > ... > y$.

Corollary 2:

When $y' > y \geq 1$,

$$a \to a \to ... \to a \to y' > a \to a \to ... \to a \to y$$

(with n+1 entries in chain on each side, first n entries contain 'a'),

for all $a \geq 3$, $z \geq 1$, $n \geq 1$.

This is the same as Corollary 1 but with $z = 1$, which means that the final entries of the chains on both sides of the inequality can be removed under the rules for Conway's Chained Arrow Notation.

With both Lemmas proven, I am ready for the main part of the proof.

Main Proof:

$$\{a, b, c, d\} - 1 > a \to a \to ... \to a \to (b-1) \to (c+1)$$

(with d+2 entries in chain, first d entries contain 'a'),

for all $a \geq 3$, $b \geq 2$, $c \geq 1$, $d \geq 2$.

This also involves proof by induction.

When $a \geq 3$, $b = 2$, $c = 1$, $d = 2$,

$$\{a, 2, 1, 2\} - 1 = \{a, a, \{a, 1, 1, 2\}\} - 1$$

$= \{a, a, a\} - 1$
$= (a \to a \to a) - 1$ (since $\{a, b, c\} = a \to b \to c$, by definition)
$> a \to a \to (a-1)$ (by Lemma 2 with $y = a-1$, $z = 1$, $n = 2$)
$> a \to a \to 1$ (by Corollary 2, since $a-1 > 1$)
$= a \to a$
$= a \to a \to 1 \to 2$,

this holds true.

Assuming that this holds true for $a \geq 3$, $b = k$, $c = 1$, $d = 2$,
when $a \geq 3$, $b = k+1$, $c = 1$, $d = 2$,

$\{a, k+1, 1, 2\} - 1 = \{a, a, \{a, k, 1, 2\}\} - 1$
$= (a \to a \to \{a, k, 1, 2\}) - 1$
(since $\{a, b, c\} = a \to b \to c$, by definition)
$> a \to a \to (\{a, k, 1, 2\} - 1)$
(by Lemma 2 with $y = \{a, k, 1, 2\} - 1$, $z = 1$, $n = 2$)
$> a \to a \to (a \to a \to (k-1) \to 2)$
(by Corollary 2, since $\{a, k, 1, 2\} - 1 > a \to a \to (k-1) \to 2$)
$= a \to a \to k \to 2$,

this holds true. So, this holds true for $a \geq 3$, $b \geq 2$, $c = 1$, $d = 2$.

Assuming that this holds true for $a \geq 3$, $b \geq 2$, $c = k$, $d = 2$,
when $a \geq 3$, $b = 2$, $c = k+1$, $d = 2$,

$\{a, 2, k+1, 2\} - 1 = \{a, \{a, 1, k+1, 2\}, k, 2\} - 1$
$= \{a, a, k, 2\} - 1$
$> a \to a \to (a-1) \to (k+1)$
$> a \to a \to 1 \to (k+1)$ (by Corollary 1, since $a-1 > 1$)
$= a \to a$
$= a \to a \to 1 \to (k+2)$,

this holds true, and assuming that this also holds true for $a \geq 3$, $b = m$, $c = k+1$, $d = 2$,
when $a \geq 3$, $b = m+1$, $c = k+1$, $d = 2$,

$\{a, m+1, k+1, 2\} - 1 = \{a, \{a, m, k+1, 2\}, k, 2\} - 1$
$> a \to a \to (\{a, m, k+1, 2\} - 1) \to (k+1)$
$> a \to a \to (a \to a \to (m-1) \to (k+2)) \to (k+1)$
(by Corollary 1, since $\{a, m, k+1, 2\} - 1 > a \to a \to (m-1) \to (k+2)$)
$= a \to a \to m \to (k+2)$,

this holds true. So, this holds true for $a \geq 3$, $b \geq 2$, $c = k+1$, $d = 2$, and therefore, this holds true for $a \geq 3$, $b \geq 2$, $c \geq 1$, $d = 2$.

Assuming that this holds true for $a \geq 3$, $b \geq 2$, $c \geq 1$, $d = k$,
when $a \geq 3$, $b = 2$, $c = 1$, $d = k+1$,

$\{a, 2, 1, k+1\} - 1 = \{a, a, \{a, 1, 1, k+1\}, k\} - 1$
$= \{a, a, a, k\} - 1$
$> a \to ... \to a \to (a-1) \to (a+1)$
(with k+2 entries in chain and ‘$a \to ... \to a$’ represents k ‘a’s)
$= a \to ... \to a \to (a \to ... \to a \to (a-2) \to (a+1)) \to a$
$\geq a \to ... \to a \to a \to a$
(by Corollary 1, since $a \to ... \to a \to (a-2) \to (a+1) \geq a$
by Lemma 1)

> a → ... → a → a → 1

(by Corollary 2, since a > 1)

= a → ... → a → a

= a → ... → a → a → 1 → 2

(with k+3 entries in chain),

this holds true, and assuming that this also holds true for a ≥ 3, b = m, c = 1, d = k+1,
when a ≥ 3, b = m+1, c = 1, d = k+1,

{a, m+1, 1, k+1} - 1 = {a, a, {a, m, 1, k+1}, k} - 1

> a → ... → a → (a-1) → ({a, m, 1, k+1} + 1)

(with k+2 entries in chain and 'a → ... → a' represents k 'a's)

= a → ... → a → (a → ... → a → (a-2) → ({a, m, 1, k+1} + 1)) → {a, m, 1, k+1}

≥ a → ... → a → a → {a, m, 1, k+1}

(by Corollary 1, since a → ... → a → (a-2) → ({a, m, 1, k+1} + 1) ≥ a by Lemma 1)

> a → ... → a → a → (a → ... → a → a → (m-1) → 2)

(by Corollary 2, since {a, m, 1, k+1} > a → ... → a → a → (m-1) → 2)

= a → ... → a → a → m → 2

(with k+3 entries in chain),

this holds true. So, this holds true for a ≥ 3, b ≥ 2, c = 1, d = k+1.

Assuming that this also holds true for a ≥ 3, b ≥ 2, c = n, d = k+1 (as well as a ≥ 3, b ≥ 2, c ≥ 1, d = k),
when a ≥ 3, b = 2, c = n+1, d = k+1,

{a, 2, n+1, k+1} - 1 = {a, {a, 1, n+1, k+1}, n, k+1} - 1

= {a, a, n, k+1} - 1

> a → ... → a → (a-1) → (n+1)

(with k+3 entries in chain and 'a → ... → a' represents k+1 'a's)

> a → ... → a → 1 → (n+1)

(by Corollary 1, since a-1 > 1)

= a → ... → a

= a → ... → a → 1 → (n+2)

(with k+3 entries in chain),

this holds true, and assuming that this also holds true for a ≥ 3, b = m, c = n+1, d = k+1,
when a ≥ 3, b = m+1, c = n+1, d = k+1,

{a, m+1, n+1, k+1} - 1 = {a, {a, m, n+1, k+1}, n, k+1} - 1

> a → ... → a → ({a, m, n+1, k+1} - 1) → (n+1)

(with k+3 entries in chain, 'a → ... → a' representing k+1 'a's)

> a → ... → a → (a → ... → a → (m-1) → (n+2)) → (n+1)

(by Corollary 1, since {a, m, n+1, k+1} - 1 > a → ... → a → (m-1) → (n+2))

= a → ... → a → m → (n+2)

(with k+3 entries in chain),

this holds true. So, this holds true for a ≥ 3, b ≥ 2, c = n+1, d = k+1, which means that this holds true for a ≥ 3, b ≥ 2, c ≥ 1, d = k+1.

Therefore, the inequality

{a, b, c, d} > a → a → ... → a → (b-1) → (c+1)

(with d+2 entries in chain, first d entries contain 'a')

holds true for all a ≥ 3, b ≥ 2, c ≥ 1, d ≥ 2 and I have completed the proof.

For example,

$\{3, 3, 1, 2\} > 3 \to 3 \to 2 \to 2,$
$\{3, 3, 2, 2\} > 3 \to 3 \to 2 \to 3,$
$\{3, 4, 2, 2\} > 3 \to 3 \to 3 \to 3,$
$\{3, 3, 1, 3\} > 3 \to 3 \to 3 \to 2 \to 2,$
$\{3, 3, 2, 3\} > 3 \to 3 \to 3 \to 2 \to 3,$
$\{3, 4, 2, 3\} > 3 \to 3 \to 3 \to 3 \to 3,$
$\{3, 3, 3, 3\} > 3 \to 3 \to 3 \to 2 \to 4,$
$\{3, 3, 1, 4\} > 3 \to 3 \to 3 \to 3 \to 2 \to 2,$
$\{4, 4, 4, 4\} > 4 \to 4 \to 4 \to 4 \to 3 \to 5,$
$\{3, 3, 1, 5\} > 3 \to 3 \to 3 \to 3 \to 3 \to 2 \to 2,$
$\{3, 3, 1, 10\} > 3 \to 3 \to 3 \to 3 \to 3 \to 3 \to 3 \to 3 \to 3 \to 3 \to 2 \to 2.$

Another result from the proof (especially for large d) is that (for $a \geq 3$)

$\{a, a, a, d\} > a \to a \to ... \to a \to (a-1) \to (a+1)$
(with d+2 entries in chain, first d entries contain ‘a’)
$= a \to a \to ... \to a \to (a \to a \to ... \to a \to (a-2) \to (a+1)) \to a$
$\geq a \to a \to ... \to a \to a \to a$
(by Corollary 1, since $a \to a \to ... \to a \to (a-2) \to (a+1) \geq a$ by Lemma 1),

so, when $a \geq 3$,

$\{a, a, a, d\} > a \to a \to ... \to a$ (with d+2 entries of ‘a’ in chain).

So, while the final entry in an array of 3 entries determines the number of Knuth’s up-arrows, this ‘array notation’ grows so phenomenally fast that the final entry in an array of just 4 entries determines the minimum length of the Conway chain. An array of length 5 would (of course) become far too huge for Conway’s Chained Arrow Notation. For example (for $a \geq 3$), while

$\{a, 2, 1, 1, 2\} = \{a, a, a, a\}$
$> a \to a \to ... \to a$ (with a+2 entries in chain)
$= N,$
$\{a, 3, 1, 1, 2\} = \{a, a, a, \{a, 2, 1, 1, 2\}\}$
$> a \to a \to ... \to a$ (with N+2 entries in chain),

and so,

$\{a, b, 1, 1, 2\} = \{a, a, a, \{a, b-1, 1, 1, 2\}\}$
$> a \to a \to ... \to a$ (with $\{a, b-1, 1, 1, 2\} + 2$ entries in chain).

This means that while

$\{3, 2, 1, 1, 2\} = \{3, 3, 3, 3\}$
$> 3 \to 3 \to 3 \to 2 \to 4$
$> 3 \to 3 \to 3 \to 3 \to 3$ (of length 5),

the number

$\{3, 5, 1, 1, 2\} > 3 \to 3 \to 3 \to ... \to 3$
(of length $3 \to 3 \to ... \to 3$ (of length $3 \to 3 \to ... \to 3$ (of length $3 \to 3 \to ... \to 3$ (of length $3 \to 3 \to 3 \to 2 \to 4$)))),

which should be sufficient proof that Bird’s Linear Array Notation with 5 or more entries generally goes beyond Conway’s Chained Arrow Notation.

One can only try to imagine the largeness of arrays with 6 or more entries! Even

$\{3, 3, 1, 1, 1, 2\} = \{3, 3, 3, 3, \{3, 3, 3, 3, 3\}\}.$

Of course, these arrays (like the numbers in Conway's Chained Arrow Notation) can contain hundreds, thousands or millions of entries, or even much, much more than that!

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