Beyond Bird’s Nested Arrays IV

How about extending even beyond my Nested Hyper-Nested Array Notation, which has a limit ordinal of $\theta(\varepsilon_{\Omega+1})$, the Bachmann-Howard ordinal? Let me introduce to you an all-new 1-hyperseparator symbol – the black circle (●). Since the new symbol is a 1-hyperseparator (like the backslash), it requires a minimum of one pair of square brackets around it, so the smallest separator containing ● is the [1 ● 2] separator.

The $\theta(\varepsilon_{\Omega+1})$ level separator

{a, b [1 ● 2] 2} = {a ‹0 ● 2› b}

= {a ‹b ‹b ‹ ... ‹b ‹b $\backslash_b$ b› b› ... › b› b› b}

(with b pairs of angle brackets).

This is equivalent to

{a, b [1 [1 [1 [... [1 [1 $\backslash_b$ 1, 2] 2] ...] 2] 2] 2] 2} (with b pairs of square brackets),

where the separator has level $\theta(\Omega\text{^}\Omega\text{^}\Omega\text{^}...\text{^}\Omega\text{^}\omega)$ (with 2b-3 Ω’s).

Using the collapsing theta function (in single-argument form when the second argument is zero) for expressing ordinals beyond the epsilon numbers, I find that the more significant separators have ordinal levels as follows:-

[1 ● 2] has level $\theta(\varepsilon_{\Omega+1})$,
[2 ● 2] has level $\theta(\varepsilon_{\Omega+1})+1$,
[1 [1 \ 2] 2 ● 2] has level $\theta(\varepsilon_{\Omega+1})+\varepsilon_0$,
[1 [1 ● 2] 2 ● 2] has level $\theta(\varepsilon_{\Omega+1})2$,
[1 [1 [1 ● 2] 2 ● 2] 2 ● 2] has level $\theta(\varepsilon_{\Omega+1})\text{^}\theta(\varepsilon_{\Omega+1})$,
[1 \ 2 ● 2] has level $\varepsilon(\theta(\varepsilon_{\Omega+1})+1) = \theta(1, \theta(\varepsilon_{\Omega+1})+1)$,
[1 [1 $\backslash_2$ 3] 2 ● 2] has level $\Gamma(\theta(\varepsilon_{\Omega+1})+1) = \theta(\Omega, \theta(\varepsilon_{\Omega+1})+1)$,
[1 [1 $\backslash_2$ 1 $\backslash_2$ 2] 2 ● 2] has level $\theta(\Omega\text{^}\Omega, \theta(\varepsilon_{\Omega+1})+1)$,
[1 [1 [1 $\backslash_3$ 3] 2] 2 ● 2] has level $\theta(\Omega\text{^}\Omega\text{^}\Omega, \theta(\varepsilon_{\Omega+1})+1)$,
[1 ● 3] has level $\theta(\varepsilon_{\Omega+1}, 1)$

(limit ordinal of $\theta(\alpha, \theta(\varepsilon_{\Omega+1})+1) = \theta(\alpha, \theta(\theta(\varepsilon_{\Omega+1}), 0)+1)$ as $\alpha \to \varepsilon_{\Omega+1}$),

[1 ● 4] has level $\theta(\varepsilon_{\Omega+1}, 2)$,
[1 ● 1 [1 \ 2] 2] has level $\theta(\varepsilon_{\Omega+1}, \varepsilon_0)$,
[1 ● 1 [1 ● 2] 2] has level $\theta(\varepsilon_{\Omega+1}, \theta(\varepsilon_{\Omega+1}))$,
[1 ● 1 [1 ● 1 [1 ● 2] 2] 2] has level $\theta(\varepsilon_{\Omega+1}, \theta(\varepsilon_{\Omega+1}, \theta(\varepsilon_{\Omega+1})))$,
[1 ● 1 \ 2] has level $\theta(\varepsilon_{\Omega+1}+1)$,
[1 ● 1 [2 $\backslash_2$ 2] 2] has level $\theta(\varepsilon_{\Omega+1}+\omega)$,
[1 ● 1 [1 $\backslash_2$ 3] 2] has level $\theta(\varepsilon_{\Omega+1}+\Omega)$,
[1 ● 1 [1 $\backslash_2$ 1 $\backslash_2$ 2] 2] has level $\theta(\varepsilon_{\Omega+1}+\Omega\text{^}\Omega)$,
[1 ● 1 [1 [1 $\backslash_3$ 3] 2] 2] has level $\theta(\varepsilon_{\Omega+1}+\Omega\text{^}\Omega\text{^}\Omega)$,
[1 ● 1 ● 2] has level $\theta(\varepsilon_{\Omega+1}2)$,
[1 ● 1 ● 1 ● 2] has level $\theta(\varepsilon_{\Omega+1}3)$,
[1 ● 1 ● 1 ● ... ● 1 ● 2] (with n ● symbols) has level $\theta(\varepsilon_{\Omega+1}n)$.

With k ● symbols (k ≥ 1) and # representing the remainder of the array,

{a, b [1 ● 1 ● 1 ● ... ● 1 ● c #] 2} = {a ‹ 0 ● 1 ● 1 ● ... ● 1 ● c # › b}

= {a ‹ S ● S ● ... ● S ● T ● c-1 # › b},

where S = ‘b \ b \ b \ ... \ b’ (with b b’s, explained on page 3),

T = ‘b ‹b ‹b ‹ ... ‹b ‹b $\backslash_b$ b› b› ... › b› b› b’ (with b-1 pairs of angle brackets).

Introducing a new family of n-hyperseparator $\bullet_n$ symbols, with $\bullet$ = [1 $\bullet_2$ 2] and the corresponding n-hyperseparator symbol $\bullet_n$ = [1 $\bullet_{n+1}$ 2] in a similar manner to \ = [1 $\backslash_2$ 2] and $\backslash_n$ = [1 $\backslash_{n+1}$ 2] in Beyond Bird's Nested Arrays III, I find that

[1 [2 $\bullet_2$ 2] 2] has level $\theta(\varepsilon_{\Omega+1}\omega)$,
[1 [1 $\bullet$ 2 $\bullet_2$ 2] 2] has level $\theta(\varepsilon_{\Omega+1}\theta(\varepsilon_{\Omega+1}))$,
[1 [1 [1 $\bullet$ 2 $\bullet_2$ 2] 2 $\bullet_2$ 2] 2] has level $\theta(\varepsilon_{\Omega+1}\theta(\varepsilon_{\Omega+1}\theta(\varepsilon_{\Omega+1})))$,
[1 [1 $\backslash_2$ 2 $\bullet_2$ 2] 2] has level $\theta(\varepsilon_{\Omega+1}\Omega)$,
[1 [1 $\backslash_2$ 1 $\backslash_2$ 2 $\bullet_2$ 2] 2] has level $\theta(\varepsilon_{\Omega+1}(\Omega\text{^}\Omega))$,
[1 [1 [1 $\backslash_3$ 3] 2 $\bullet_2$ 2] 2] has level $\theta(\varepsilon_{\Omega+1}(\Omega\text{^}\Omega\text{^}\Omega))$,
[1 [1 $\bullet_2$ 3] 2] has level $\theta(\varepsilon_{\Omega+1}\text{^}2)$,
[1 [1 $\bullet_2$ 4] 2] has level $\theta(\varepsilon_{\Omega+1}\text{^}3)$,
[1 [1 $\bullet_2$ 1 \ 2] 2] has level $\theta(\varepsilon_{\Omega+1}\text{^}\varepsilon_0)$,
[1 [1 $\bullet_2$ 1 $\bullet$ 2] 2] has level $\theta(\varepsilon_{\Omega+1}\text{^}\theta(\varepsilon_{\Omega+1}))$,
[1 [1 $\bullet_2$ 1 [1 $\bullet_2$ 1 $\bullet$ 2] 2] 2] has level $\theta(\varepsilon_{\Omega+1}\text{^}\theta(\varepsilon_{\Omega+1}\text{^}\theta(\varepsilon_{\Omega+1})))$,
[1 [1 $\bullet_2$ 1 $\backslash_2$ 2] 2] has level $\theta(\varepsilon_{\Omega+1}\text{^}\Omega)$,
[1 [1 $\bullet_2$ 1 $\backslash_2$ 3] 2] has level $\theta(\varepsilon_{\Omega+1}\text{^}(\Omega2))$,
[1 [1 $\bullet_2$ 1 $\backslash_2$ 1 $\backslash_2$ 2] 2] has level $\theta(\varepsilon_{\Omega+1}\text{^}\Omega\text{^}2)$,
[1 [1 $\bullet_2$ 1 [2 $\backslash_3$ 2] 2] 2] has level $\theta(\varepsilon_{\Omega+1}\text{^}\Omega\text{^}\omega)$,
[1 [1 $\bullet_2$ 1 [1 $\backslash_3$ 3] 2] 2] has level $\theta(\varepsilon_{\Omega+1}\text{^}\Omega\text{^}\Omega)$,
[1 [1 $\bullet_2$ 1 [1 $\backslash_3$ 1 $\backslash_3$ 2] 2] 2] has level $\theta(\varepsilon_{\Omega+1}\text{^}\Omega\text{^}\Omega\text{^}\Omega)$,
[1 [1 $\bullet_2$ 1 [1 [1 $\backslash_4$ 3] 2] 2] 2] has level $\theta(\varepsilon_{\Omega+1}\text{^}\Omega\text{^}\Omega\text{^}\Omega\text{^}\Omega)$,
[1 [1 $\bullet_2$ 1 $\bullet_2$ 2] 2] has level $\theta(\varepsilon_{\Omega+1}\text{^}\varepsilon_{\Omega+1})$,
[1 [1 $\bullet_2$ 1 $\bullet_2$ 3] 2] has level $\theta(\varepsilon_{\Omega+1}\text{^}(\varepsilon_{\Omega+1}2))$,
[1 [1 $\bullet_2$ 1 $\bullet_2$ 1 $\bullet_2$ 2] 2] has level $\theta(\varepsilon_{\Omega+1}\text{^}\varepsilon_{\Omega+1}\text{^}2)$,
[1 [1 $\bullet_2$ 1 $\bullet_2$ 1 $\bullet_2$ 1 $\bullet_2$ 2] 2] has level $\theta(\varepsilon_{\Omega+1}\text{^}\varepsilon_{\Omega+1}\text{^}3)$,
[1 [1 [2 $\bullet_3$ 2] 2] 2] has level $\theta(\varepsilon_{\Omega+1}\text{^}\varepsilon_{\Omega+1}\text{^}\omega)$,
[1 [1 [1 $\bullet$ 2 $\bullet_3$ 2] 2] 2] has level $\theta(\varepsilon_{\Omega+1}\text{^}\varepsilon_{\Omega+1}\text{^}\theta(\varepsilon_{\Omega+1}))$,
[1 [1 [1 $\backslash_3$ 2 $\bullet_3$ 2] 2] 2] has level $\theta(\varepsilon_{\Omega+1}\text{^}\varepsilon_{\Omega+1}\text{^}\Omega)$,
[1 [1 [1 $\backslash_3$ 1 $\backslash_3$ 2 $\bullet_3$ 2] 2] 2] has level $\theta(\varepsilon_{\Omega+1}\text{^}\varepsilon_{\Omega+1}\text{^}\Omega\text{^}\Omega)$,
[1 [1 [1 [1 $\backslash_4$ 3] 2 $\bullet_3$ 2] 2] 2] has level $\theta(\varepsilon_{\Omega+1}\text{^}\varepsilon_{\Omega+1}\text{^}\Omega\text{^}\Omega\text{^}\Omega)$,
[1 [1 [1 $\bullet_3$ 3] 2] 2] has level $\theta(\varepsilon_{\Omega+1}\text{^}\varepsilon_{\Omega+1}\text{^}\varepsilon_{\Omega+1})$,
[1 [1 [1 $\bullet_3$ 1 $\bullet_3$ 2] 2] 2] has level $\theta(\varepsilon_{\Omega+1}\text{^}\varepsilon_{\Omega+1}\text{^}\varepsilon_{\Omega+1}\text{^}\varepsilon_{\Omega+1})$,
[1 [1 [1 [1 $\bullet_4$ 3] 2] 2] 2] has level $\theta(\varepsilon_{\Omega+1}\text{^}\varepsilon_{\Omega+1}\text{^}\varepsilon_{\Omega+1}\text{^}\varepsilon_{\Omega+1}\text{^}\varepsilon_{\Omega+1})$,
[1 \[3] 2] has level $\theta(\varepsilon_{\Omega+2})$ ($\backslash_n$ is \[1]$_n$ and $\bullet_n$ is \[2]$_n$ – when n = 1, \ is \[1] and $\bullet$ is \[2]),
[1 \[4] 2] has level $\theta(\varepsilon_{\Omega+3})$ (by letting $\bullet_n$ and $\varepsilon_{\Omega+1}$ above be \[3]$_n$ and $\varepsilon_{\Omega+2}$ respectively),
[1 \[1, 2] 2] has level $\theta(\varepsilon_{\Omega+\omega})$,
[1 \[1, 1, 2] 2] has level $\theta(\varepsilon_{\Omega+\omega\text{^}2})$,
[1 \[1 [2] 2] 2] has level $\theta(\varepsilon_{\Omega+\omega\text{^}\omega})$,
[1 \[1 \ 2] 2] has level $\theta(\varepsilon(\Omega+\varepsilon_0))$,
[1 \[1 [1 $\backslash_2$ 3] 2] 2] has level $\theta(\varepsilon(\Omega+\Gamma_0))$,
[1 \[1 $\bullet$ 2] 2] has level $\theta(\varepsilon(\Omega+\theta(\varepsilon_{\Omega+1})))$,
[1 \[1 \[1 \ 2] 2] 2] has level $\theta(\varepsilon(\Omega+\theta(\varepsilon(\Omega+\varepsilon_0))))$,
[1 \[1 \[1 $\bullet$ 2] 2] 2] has level $\theta(\varepsilon(\Omega+\theta(\varepsilon(\Omega+\theta(\varepsilon_{\Omega+1})))))$,
[1 \[1 \[1 \[1 $\bullet$ 2] 2] 2] 2] has level $\theta(\varepsilon(\Omega+\theta(\varepsilon(\Omega+\theta(\varepsilon(\Omega+\theta(\varepsilon_{\Omega+1})))))))$.

The limit ordinal of the \[] bracket notation (with ω nested levels of \[] brackets) is $\theta(\varepsilon_{\Omega2})$.

In general, when X is an array, the n-hyperseparator \[X]$_n$ = [1 \[X]$_{n+1}$ 2]. The subscript is omitted when it is 1, since this is the lowest value. Angle Bracket Rules A5a-b (page 24 of Beyond Bird's Nested Arrays III) now apply to all backslash arrays; \ now reads \[B] and $\backslash_j$ now reads \[B]$_j$ for any array B.

When $A_1, A_2, \ldots, A_k$ are 1-hyperseparator arrays ($k \geq 0$) that contain the symbols $\backslash[H_1]_{n(1)}$, $\backslash[H_2]_{n(2)}$, ... , $\backslash[H_k]_{n(k)}$ respectively, where each H_i is the highest array within \[] brackets contained within A_i and placed within n(i) layers of square brackets (if n(i) = 1 then $[A_i] = \backslash[H_i]$); B is an array within \[] brackets and # is the remainder of the angle bracket array (including any 2- or higher order hyperseparators),

‘a ‹ 0 [A_1] 1 [A_2] ... 1 [A_k] 1 \[B] c # › b’

= ‘a ‹ b ‹A_1’› b [A_1] b ‹A_2’› b [A_2] ... b ‹A_k’› b [A_k] S \[B] c-1 # › b’,

where S = ‘b ‹b ‹b ‹ ... ‹b ‹b $\backslash[C]_b$ b› b› ... › b› b› b’ (with b-1 pairs of angle brackets, B ≠ ‘1 #*’)

= ‘b \‹C› b’ (B = ‘1 #*’, where #* is a non-empty string)

and A_1’, A_2’, ... , A_k’, C are arrays that are identical to $A_1, A_2, \ldots, A_k$, B respectively except that the first entries of each have been reduced by 1 (see below for B = ‘1 #*’ and for backslash angle bracket arrays). When n(i) = 1 for some $1 \leq i \leq k$ then $[A_i] = \backslash[H_i]$ and ‘b ‹A_i’› b’ = ‘b \‹H_i’› b’. It is worth noting that $\backslash[H_1] \geq \backslash[H_2] \geq \ldots \geq \backslash[H_k] \geq \backslash[B]$.

The S string is different when \[B] (above) begins with 1, since $\backslash[C]_b$ cannot begin with 0 (as it is not an angle bracket array). Instead, C is placed in a backslash angle bracket array, which works in a similar way to a normal angle bracket array except that it creates backslash square (\[]) and backslash angle bracket (\‹›) arrays in their places. If \[B] = \[1 #*] then S = ‘b \‹0 #*› b’, which evaluates further in the examples below:

If \[B] = \[1, d #*] then S = ‘b \‹b, d-1 #*› b’,

if \[B] = \[1, 1, ... , 1, d #*] (with m 1’s) then S = ‘b \‹ b, b, ... , b, d-1 #* › b’ (with m b’s in \‹›),

if \[B] = \[1 [2] d #*] then S = ‘b \‹ b ‹1› b [2] d-1 #* › b’,

if \[B] = \[1 [X] d #*] then S = ‘b \‹ b ‹X’› b [X] d-1 #* › b’,

if \[B] = \[1 \ d #* $\#_H$] then S = ‘b \‹R_b $\#_H$› b’ with R_n = ‘b ‹R_{n-1}› b \ d-1 #*’ and R_1 = ‘0’,

if \[B] = \[1 [1 $\backslash_2$ m] d #* $\#_H$] (m ≥ 3) then S = ‘b \‹ b ‹R_b› b [1 $\backslash_2$ m] d-1 #* $\#_H$ › b’
with R_n = ‘b ‹R_{n-1}› b [1 $\backslash_2$ m] d-1 #* $\backslash_2$ m-1’ and R_1 = ‘0’,

if \[B] = \[1 \[m #**] d #*] (m ≥ 2) then S = ‘b \‹ T \[m #**] d-1 #* › b’
with T = ‘b ‹b ‹b ‹ ... ‹b ‹b $\backslash[m\text{-}1\ \#{*}{*}]_b$ b› b› ... › b› b› b’ (with b-1 pairs of angle brackets),

if \[B] = \[1 \[1, e #**] d #*] (m ≥ 2) then S = ‘b \‹ T \[1, e #**] d-1 #* › b’
with T = ‘b \‹0, e #**› b’ = ‘b \‹b, e-1 #**› b’,

where #* and #** are the rest of their respective backslash arrays, $\#_H$ is either an empty string or begins with the first 2- or higher hyperseparator in the backslash array, X does not begin with 1 and a 2-hyperseparator (e.g. cannot be \ or [1 $\backslash_2$ m]) and X’ is identical to X except that the first entry is reduced by 1. If X begins with 1 and a 2-hyperseparator, we would need to use Angle Bracket Rules A5a-e (pages 24-25 of Beyond Bird’s Nested Arrays III) in order to find S. (Rules A5c-d entail repeating Rules A5a-e.)

The backslash angle bracket array

‘a \‹n #› b’ = ‘a \‹n-1 #› b \[n #] a \‹n-1 #› b \[n #] ... \[n #] a \‹n-1 #› b’
(with b ‘a \‹n-1 #› b’ strings),

mirroring Rule A6 for normal angle bracket arrays. Since the k ● symbols on the bottom of the first page of this document are shorthand for \[2] symbols, each of the S strings making up the array {a ‹ S ● S ● ... ● S ● T ● c-1 # › b} are equal to

‘b \‹1› b’ = ‘b \‹0› b \[1] b \‹0› b \[1] ... \[1] b \‹0› b’ (with b ‘b \‹0› b’ strings)
= ‘b \ b \ b \ ... \ b’ (with b b’s),

as the [1]’s are removed and ‘b \‹0› b’ = ‘b’, just as ‘b ‹0 #› b’ = ‘b’ when # is a 2- or higher order hyperseparator (or omitted).

When \[B] is \[1], the [1] is dropped, # splits into two and the string replacement equation becomes

‘a ‹ 0 [A_1] 1 [A_2] ... 1 [A_k] 1 \ c # $\#_H$ › b’ = ‘a ‹R_b $\#_H$› b’,

where R_n = ‘b ‹A_1’› b [A_1] b ‹A_2’› b [A_2] ... b ‹A_k’› b [A_k] b ‹R_{n-1}› b \ c-1 #’,
R_1 = ‘0’
and $\#_H$ is either an empty string or begins with a 2- or higher hyperseparator (not contained in the base of #). This is similar to Rule A5a.

When B is a 1-hyperseparator array that contains the m-hyperseparator \[H]$_m$, where H is the highest array within \[] brackets contained within B and placed within m layers of square brackets (m ≥ 2) as follows:

[B] = [B_1],
[B_i] = [1 [$A_{i+1,1}$] 1 [$A_{i+1,2}$] ... 1 [$A_{i+1,k(i+1)}$] 1 [B_{i+1}] c_{i+1} $\#_{i+1}$] (1 ≤ i < m-1),
[B_{m-1}] = [1 [$A_{m,1}$] 1 [$A_{m,2}$] ... 1 [$A_{m,k(m)}$] 1 \[H]$_m$ c_m $\#_m$],

where k(i) ≥ 0, c_i ≥ 2 and each of [$A_{i,j}$] and [B_i] is an i-hyperseparator (any of the [$A_{i,j}$] may be replaced by a \[$A_{i,j}$]$_i$ symbol, which would mean that ‘b ‹$A_{i,j}$’› b’ is rewritten as ‘b \‹$A_{i,j}$’›$_i$ b’, see below), the string replacement equation becomes

‘a ‹ 0 [A_1] 1 [A_2] ... 1 [A_k] 1 [B] c # › b’
= ‘a ‹ b ‹A_1’› b [A_1] b ‹A_2’› b [A_2] ... b ‹A_k’› b [A_k] b ‹S_1› b [B] c-1 # › b’,

where S_i = ‘b ‹$A_{i+1,1}$’› b [$A_{i+1,1}$] b ‹$A_{i+1,2}$’› b [$A_{i+1,2}$] ... b ‹$A_{i+1,k(i+1)}$’› b [$A_{i+1,k(i+1)}$] b ‹S_{i+1}› b [B_{i+1}] c_{i+1}-1 $\#_{i+1}$’
(1 ≤ i < m-1),
S_{m-1} = ‘b ‹$A_{m,1}$’› b [$A_{m,1}$] b ‹$A_{m,2}$’› b [$A_{m,2}$] ... b ‹$A_{m,k(m)}$’› b [$A_{m,k(m)}$] T \[H]$_m$ c_m-1 $\#_m$’,
T = ‘b ‹b ‹b ‹ ... ‹b ‹b \[C]$_{m+b-1}$ b› b› ... › b› b› b’ (with b-1 pairs of angle brackets, H ≠ ‘1 #*’)
= ‘b \‹C›$_m$ b’ (H = ‘1 #*’, where #* is a non-empty string)

and $A_{i,j}$’ and C are identical to $A_{i,j}$ and H respectively except that the first entries of each have been reduced by 1.

Backslash angle brackets with subscripts work in a similar way to those without subscripts except that the square and angle bracket arrays created in their places would themselves contain subscripts, for example,

‘3 \‹k #›$_n$ 3’ = ‘3 \‹k-1 #›$_n$ 3 \[k #]$_n$ 3 \‹k-1 #›$_n$ 3 \[k #]$_n$ 3 \‹k-1 #›$_n$ 3’ (k ≥ 1),
‘a \‹0 [m #*] k #›$_n$ b’ = ‘a \‹b ‹m-1 #*› b [m #*] k-1 #›$_n$ b’ (k ≥ 2),

where #* does not begin with a 2- or higher order hyperseparator when m = 1. If the # in the string ‘a \‹0 #›$_n$ b’ does begin with a 2- or higher hyperseparator (or is empty) then ‘a \‹0 #›$_n$ b’ = ‘a’.

If \[H]$_m$ is \[1]$_m$, the [1] is omitted, and we would make m-1 applications of Rule A5d, followed by Rule A5b.

In order to proceed further, it is time to introduce another all-new special symbol – the double backslash (\\), which requires a minimum of two pairs of square brackets enclosing it. By rewriting the n-hyperseparator \[X]$_n$ (for an array X) as [X \\ 2]$_n$ (so that \$_n$ is [1 \\ 2]$_n$ and ●$_n$ is [2 \\ 2]$_n$), I obtain

[1 [1 \\ 3] 2] has level $\theta(\varepsilon_{\Omega 2})$,
[1 [1 \\ 3] 1 [1 \\ 3] 2] has level $\theta(\varepsilon_{\Omega 2}2)$,
[1 [2 [1 \\ 3]$_2$ 2] 2] has level $\theta(\varepsilon_{\Omega 2}\omega)$,
[1 [1 \ 2 [1 \\ 3]$_2$ 2] 2] has level $\theta(\varepsilon_{\Omega 2}\varepsilon_0)$,
[1 [1 [1 \\ 3] 2 [1 \\ 3]$_2$ 2] 2] has level $\theta(\varepsilon_{\Omega 2}\theta(\varepsilon_{\Omega 2}))$,
[1 [1 [1 [1 \\ 3] 2 [1 \\ 3]$_2$ 2] 2 [1 \\ 3]$_2$ 2] 2] has level $\theta(\varepsilon_{\Omega 2}\theta(\varepsilon_{\Omega 2}\theta(\varepsilon_{\Omega 2})))$,
[1 [1 \$_2$ 2 [1 \\ 3]$_2$ 2] 2] has level $\theta(\varepsilon_{\Omega 2}\Omega)$,
[1 [1 ●$_2$ 2 [1 \\ 3]$_2$ 2] 2] has level $\theta(\varepsilon_{\Omega 2}\varepsilon_{\Omega+1})$,
[1 [1 [3 \\ 2]$_2$ 2 [1 \\ 3]$_2$ 2] 2] has level $\theta(\varepsilon_{\Omega 2}\varepsilon_{\Omega+2})$,
[1 [1 [1 ● 2 \\ 2]$_2$ 2 [1 \\ 3]$_2$ 2] 2] has level $\theta(\varepsilon_{\Omega 2}\varepsilon(\Omega+\theta(\varepsilon_{\Omega+1})))$,

[1 [1 [1 [1 [1 ● 2 \\ 2]$_2$ 2 [1 \\ 3]$_2$ 2] 2 \\ 2]$_2$ 2 [1 \\ 3]$_2$ 2] 2] has level
θ(ε$_{Ω2}$ε(Ω+θ(ε$_{Ω2}$ε(Ω+θ(ε$_{Ω+1}$))))),
[1 [1 [1 \\ 3]$_2$ 3] 2] has level θ(ε$_{Ω2}$^2),
[1 [1 [1 \\ 3]$_2$ 4] 2] has level θ(ε$_{Ω2}$^3),
[1 [1 [1 \\ 3]$_2$ 1 \ 2] 2] has level θ(ε$_{Ω2}^{ε}_0$),
[1 [1 [1 \\ 3]$_2$ 1 ● 2] 2] has level θ(ε$_{Ω2}$^θ(ε$_{Ω+1}$)),
[1 [1 [1 \\ 3]$_2$ 1 [1 \\ 3] 2] 2] has level θ(ε$_{Ω2}$^θ(ε$_{Ω2}$)),
[1 [1 [1 \\ 3]$_2$ 1 [1 [1 \\ 3]$_2$ 1 [1 \\ 3] 2] 2] 2] has level θ(ε$_{Ω2}$^θ(ε$_{Ω2}$^θ(ε$_{Ω2}$))),
[1 [1 [1 \\ 3]$_2$ 1 \$_2$ 2] 2] has level θ(ε$_{Ω2}$^Ω),
[1 [1 [1 \\ 3]$_2$ 1 [1 \$_3$ 3] 2] 2] has level θ(ε$_{Ω2}$^Ω^Ω),
[1 [1 [1 \\ 3]$_2$ 1 ●$_2$ 2] 2] has level θ(ε$_{Ω2}^{ε}_{Ω+1}$),
[1 [1 [1 \\ 3]$_2$ 1 [3 \\ 2]$_2$ 2] 2] has level θ(ε$_{Ω2}^{ε}_{Ω+2}$),
[1 [1 [1 \\ 3]$_2$ 1 [1 \\ 3]$_2$ 2] 2] has level θ(ε$_{Ω2}^{ε}_{Ω2}$),
[1 [1 [1 \\ 3]$_2$ 1 [1 \\ 3]$_2$ 1 [1 \\ 3]$_2$ 2] 2] has level θ(ε$_{Ω2}^{ε}_{Ω2}$^2),
[1 [1 [2 [1 \\ 3]$_3$ 2] 2] 2] has level θ(ε$_{Ω2}^{ε}_{Ω2}$^ω),
[1 [1 [1 \$_3$ 2 [1 \\ 3]$_3$ 2] 2] 2] has level θ(ε$_{Ω2}^{ε}_{Ω2}$^Ω),
[1 [1 [1 ●$_3$ 2 [1 \\ 3]$_3$ 2] 2] 2] has level θ(ε$_{Ω2}^{ε}_{Ω2}^{ε}_{Ω+1}$),
[1 [1 [1 [3 \\ 2]$_3$ 2 [1 \\ 3]$_3$ 2] 2] 2] has level θ(ε$_{Ω2}^{ε}_{Ω2}^{ε}_{Ω+2}$),
[1 [1 [1 [1 ● 2 \\ 2]$_3$ 2 [1 \\ 3]$_3$ 2] 2] 2] has level θ(ε$_{Ω2}^{ε}_{Ω2}$^ε(Ω+θ(ε$_{Ω+1}$))),
[1 [1 [1 [1 [1 [1 [1 ● 2 \\ 2]$_3$ 2 [1 \\ 3]$_3$ 2] 2] 2 \\ 2]$_3$ 2 [1 \\ 3]$_3$ 2] 2] 2] has level
θ(ε$_{Ω2}^{ε}_{Ω2}$^ε(Ω+θ(ε$_{Ω2}^{ε}_{Ω2}$^ε(Ω+θ(ε$_{Ω+1}$))))),
[1 [1 [1 [1 \\ 3]$_3$ 3] 2] 2] has level θ(ε$_{Ω2}^{ε}_{Ω2}^{ε}_{Ω2}$),
[1 [1 [1 [1 \\ 3]$_3$ 1 [1 \\ 3]$_3$ 2] 2] 2] has level θ(ε$_{Ω2}^{ε}_{Ω2}^{ε}_{Ω2}^{ε}_{Ω2}$),
[1 [1 [1 [1 [1 \\ 3]$_4$ 3] 2] 2] 2] has level θ(ε$_{Ω2}^{ε}_{Ω2}^{ε}_{Ω2}^{ε}_{Ω2}^{ε}_{Ω2}$),
[1 [2 \\ 3] 2] has level θ(ε$_{Ω2+1}$),
[1 [3 \\ 3] 2] has level θ(ε$_{Ω2+2}$),
[1 [1 \ 2 \\ 3] 2] has level θ(ε(Ω2+ε$_0$)),
[1 [1 [1 \\ 3] 2 \\ 3] 2] has level θ(ε(Ω2+θ(ε$_{Ω2}$))),
[1 [1 [1 [1 \\ 3] 2 \\ 3] 2 \\ 3] 2] has level θ(ε(Ω2+θ(ε(Ω2+θ(ε$_{Ω2}$))))).

[1 [1 \\ 4] 2] has level θ(ε$_{Ω3}$),
[1 [1 \\ 5] 2] has level θ(ε$_{Ω4}$),
[1 [1 \\ 1 \ 2] 2] has level θ(ε(Ωε$_0$)),
[1 [1 \\ 1 ● 2] 2] has level θ(ε(Ωθ(ε$_{Ω+1}$))),
[1 [1 \\ 1 [1 \\ 1 ● 2] 2] 2] has level θ(ε(Ωθ(ε(Ωθ(ε$_{Ω+1}$))))),
[1 [1 \\ 1 [1 \\ 1 [1 \\ 1 ● 2] 2] 2] 2] has level θ(ε(Ωθ(ε(Ωθ(ε(Ωθ(ε$_{Ω+1}$))))))),
[1 [1 \\ 1 \\ 2] 2] has level θ(ε$_{Ω\text{^}2}$),
[1 [1 \\ 1 \\ 1 \\ 2] 2] has level θ(ε$_{Ω\text{^}3}$),
[1 [1 [2 \\$_2$ 2] 2] 2] has level θ(ε$_{Ω\text{^}ω}$)   (\\ = [1 \\$_2$ 2] just as \ = [1 \$_2$ 2]),
[1 [1 [1 \ 2 \\$_2$ 2] 2] 2] has level θ(ε(Ω^ε$_0$)),
[1 [1 [1 ● 2 \\$_2$ 2] 2] 2] has level θ(ε(Ω^θ(ε$_{Ω+1}$))),
[1 [1 [1 [1 [1 ● 2 \\$_2$ 2] 2] 2 \\$_2$ 2] 2] 2] has level θ(ε(Ω^θ(ε(Ω^θ(ε$_{Ω+1}$))))),
[1 [1 [1 [1 [1 [1 [1 ● 2 \\$_2$ 2] 2] 2 \\$_2$ 2] 2] 2 \\$_2$ 2] 2] 2] has level θ(ε(Ω^θ(ε(Ω^θ(ε(Ω^θ(ε$_{Ω+1}$))))))).

[1 [1 [1 \\$_2$ 3] 2] 2] has level θ(ε$_{Ω\text{^}Ω}$),
[1 [1 \\ 2 [1 \\$_2$ 3] 2] 2] has level θ(ε$_{Ω\text{^}Ω+Ω}$),
[1 [1 [2 \\$_2$ 2] 2 [1 \\$_2$ 3] 2] 2] has level θ(ε$_{Ω\text{^}Ω+Ω\text{^}ω}$),
[1 [1 [1 \\$_2$ 3] 3] 2] has level θ(ε$_{(Ω\text{^}Ω)2}$),
[1 [1 [1 \\$_2$ 3] 1 \\ 2] 2] has level θ(ε$_{Ω\text{^}(Ω+1)}$),

$[1\ [1\ [1 \backslash\backslash_2\ 3]\ 1\ [2 \backslash\backslash_2\ 2]\ 2]\ 2]$ has level $\theta(\varepsilon_{\Omega\wedge(\Omega+\omega)})$,
$[1\ [1\ [1 \backslash\backslash_2\ 3]\ 1\ [1 \backslash\backslash_2\ 3]\ 2]\ 2]$ has level $\theta(\varepsilon_{\Omega\wedge(\Omega 2)})$,
$[1\ [1\ [2 \backslash\backslash_2\ 3]\ 2]\ 2]$ has level $\theta(\varepsilon_{\Omega\wedge(\Omega\omega)})$,
$[1\ [1\ [1 \backslash 2 \backslash\backslash_2\ 3]\ 2]\ 2]$ has level $\theta(\varepsilon(\Omega\wedge(\Omega\varepsilon_0)))$,
$[1\ [1\ [1\ [1\ [1 \backslash\backslash_2\ 3]\ 2]\ 2 \backslash\backslash_2\ 3]\ 2]\ 2]$ has level $\theta(\varepsilon(\Omega\wedge(\Omega\theta(\varepsilon_{\Omega\wedge\Omega}))))$,
$[1\ [1\ [1 \backslash\backslash_2\ 4]\ 2]\ 2]$ has level $\theta(\varepsilon_{\Omega\wedge\Omega\wedge 2})$,
$[1\ [1\ [1 \backslash\backslash_2\ 5]\ 2]\ 2]$ has level $\theta(\varepsilon_{\Omega\wedge\Omega\wedge 3})$,
$[1\ [1\ [1 \backslash\backslash_2\ 1 \backslash 2]\ 2]\ 2]$ has level $\theta(\varepsilon(\Omega\wedge\Omega\wedge\varepsilon_0))$,
$[1\ [1\ [1 \backslash\backslash_2\ 1\ [1\ [1 \backslash\backslash_2\ 3]\ 2]\ 2]\ 2]\ 2]$ has level $\theta(\varepsilon(\Omega\wedge\Omega\wedge\theta(\varepsilon_{\Omega\wedge\Omega})))$.

$[1\ [1\ [1 \backslash\backslash_2\ 1 \backslash\backslash_2\ 2]\ 2]\ 2]$ has level $\theta(\varepsilon_{\Omega\wedge\Omega\wedge\Omega})$,
$[1\ [1\ [1 \backslash\backslash_2\ 1 \backslash\backslash_2\ 1 \backslash\backslash_2\ 2]\ 2]\ 2]$ has level $\theta(\varepsilon_{\Omega\wedge\Omega\wedge\Omega\wedge 2})$,
$[1\ [1\ [1\ [2 \backslash\backslash_3\ 2]\ 2]\ 2]\ 2]$ has level $\theta(\varepsilon_{\Omega\wedge\Omega\wedge\Omega\wedge\omega})$ (in general, $\backslash\backslash_n = [1 \backslash\backslash_{n+1}\ 2]$),
$[1\ [1\ [1\ [1 \backslash 2 \backslash\backslash_3\ 2]\ 2]\ 2]\ 2]$ has level $\theta(\varepsilon(\Omega\wedge\Omega\wedge\Omega\wedge\varepsilon_0))$,
$[1\ [1\ [1\ [1\ [1\ [1 \backslash\backslash_2\ 1 \backslash\backslash_2\ 2]\ 2]\ 2 \backslash\backslash_3\ 2]\ 2]\ 2]\ 2]$ has level $\theta(\varepsilon(\Omega\wedge\Omega\wedge\Omega\wedge\theta(\varepsilon_{\Omega\wedge\Omega\wedge\Omega})))$,
$[1\ [1\ [1\ [1 \backslash\backslash_3\ 3]\ 2]\ 2]\ 2]$ has level $\theta(\varepsilon_{\Omega\wedge\Omega\wedge\Omega\wedge\Omega})$,
$[1\ [1\ [1\ [1 \backslash\backslash_3\ 1 \backslash\backslash_3\ 2]\ 2]\ 2]\ 2]$ has level $\theta(\varepsilon_{\Omega\wedge\Omega\wedge\Omega\wedge\Omega\wedge\Omega})$,
$[1\ [1\ [1\ [1\ [1 \backslash\backslash_4\ 3]\ 2]\ 2]\ 2]\ 2]$ has level $\theta(\varepsilon_{\Omega\wedge\Omega\wedge\Omega\wedge\Omega\wedge\Omega\wedge\Omega})$.

The sequence of separators starting with the last three has limit ordinal $\theta(\varepsilon(\varepsilon_{\Omega+1}))$. This is Nested Hyper-Nested Array Notation extended to the next level.

When the k ● symbols on the bottom of the first page of this document are each replaced by $[d \backslash\backslash e \#^*]$ separators (where $d \geq 2$ and $\#^*$ is the remainder of the separator array), S = ‘b ‹d-1 \\ e #*› b’ and the $\backslash_b$ symbol in the T string would be replaced by the $[d\text{-}1 \backslash\backslash e \#^*]_b$ separator. A double backslash array nested to the $\backslash\backslash_n$ level through an m-hyperseparator array ($[]_m$ brackets) evaluates in a similar fashion to a single backslash array nested to the $\backslash_{n+m}$ level. The $\backslash\backslash_n$ symbol (requiring a minimum of n+1 pairs of square brackets in a curly bracket array) is an n-hyperseparator on the second level in the extension to my Nested Hyper-Nested Array Notation, so I will regard it as an (n, 2)-hyperseparator.

The $\backslash[H_i]_{n(i)}$, $\backslash[B]$, $\backslash[C]_b$, $\backslash[H]_m$ and $\backslash[C]_{m+b-1}$ symbols on pages 3-4 (arrays within \[] brackets) can now be replaced by special separators that contain at least one (1, 2)-hyperseparator ($\backslash\backslash_n$ symbol enclosed by n-1 pairs of square brackets) in their ‘base layers’, for example, $[X \backslash\backslash d \#]_m$, $[X\ [Y \backslash\backslash_2\ e\ \#_2]\ d\ \#_1]_m$ and $[X\ [Y\ [Z \backslash\backslash_3\ f\ \#_3]\ e\ \#_2]\ d\ \#_1]_m$, where the capital letters denote strings of numbers and separators, lower case letters denote positive integers and # symbols represent the remainder of their respective separator arrays. (Similarly, the \‹C› and $\backslash\langle C\rangle_m$ symbols within \‹› brackets can now be rewritten without the preceding backslash.) The separator subscript (m) is omitted whenever it is 1, as with single or double backslash subscripts. If the $\backslash[H]_m$ symbol is rewritten as $[H\ [S]\ d\ \#^*]_m$, where [S] is the first (1, 2)-hyperseparator array within the symbol, the $\backslash[C]_{m+b-1}$ and $\backslash\langle C\rangle_m$ symbols would be rewritten as $[C\ [S]\ d\ \#^*]_{m+b-1}$ and $\langle C\ [S]\ d\ \#^*\rangle_m$ respectively.

In the Angle Bracket Rules, the term ‘(j+1)-hyperseparator’ in Rules A5c-d is now renamed ‘2- or higher order hyperseparator’ since this includes (n, 2)-hyperseparators for any n (n would be less than j). An extra subrule within Rule A5 is created as follows:-

Rule A5b* (separator $[A_{i,j}(p_{i,j})] = [d\ \#_S]_j$, where $d \geq 2$ and $\#_S$ contains at least one (1, 2)-hyperseparator in its base layer):

$S_{i,j}$ = ‘b $\langle A_{i,j}(1)'\rangle$ b $[A_{i,j}(1)]$ b $\langle A_{i,j}(2)'\rangle$ b $[A_{i,j}(2)]$ ... b $\langle A_{i,j}(p_{i,j}-1)'\rangle$ b $[A_{i,j}(p_{i,j}-1)]$ R_b $[d\ \#_S]_j$ $c_{i,j}$-1 $\#_{i,j}$’,
R_n = ‘b $\langle R_{n-1}\rangle$ b’,
R_1 = ‘b $[d\text{-}1\ \#_S]_{b+j-1}$ b’.

Note that Rule A5b* with $[A_{i,j}(p_{i,j})] = [2 \backslash\backslash 2]_j$ would mean that R_1 = ‘b $[1 \backslash\backslash 2]_{b+j-1}$ b’ = ‘b $\backslash_{b+j-1}$ b’. Setting j = 1 gives R_1 = ‘b $\backslash_b$ b’, which enables us to go beyond the Bachmann-Howard ordinal.

If the separator $[A_{i,j}(p_{i,j})] = [1\ [B_1]\ 1\ [B_2]\ \ldots\ 1\ [B_q]\ d\ \#_S]_j$, where $q \geq 1$, $d \geq 2$, each of $[B_i]$ is either a normal separator or 1-hyperseparator, and $\#_S$ contains at least one (1, 2)-hyperseparator in its ‘base layer’, Rule A5e would apply, but with the separator $[A_{i,j}(p_{i,j})]$ and the associated angle bracket array each carrying the subscript j:

$$S_{i,j} = \text{‘}b\ \langle A_{i,j}(1)' \rangle\ b\ [A_{i,j}(1)]\ b\ \langle A_{i,j}(2)' \rangle\ b\ [A_{i,j}(2)]\ \ldots\ b\ \langle A_{i,j}(p_{i,j}\text{-}1)' \rangle\ b\ [A_{i,j}(p_{i,j}\text{-}1)]$$
$$b\ \langle A_{i,j}(p_{i,j})' \rangle_j\ b\ \ [A_{i,j}(p_{i,j})]_j\ \ c_{i,j}\text{-}1\ \#_{i,j}\text{’}.$$

Any 2- or higher hyperseparator may carry a subscript. For example, in Rule A5, if the j-hyperseparator $[A_{i,j}(i^*)]$, for some $j \geq 2$ and $1 \leq i^* \leq p_{i,j}$, has the subscript j, it is written $[A_{i,j}(i^*)]_j$, and the associated angle bracket array that replaces the preceding 1 would be ‘b $\langle A_{i,j}(i^*)' \rangle_j$ b’.

Angle brackets with subscripts work in a similar way to those without them except that the square and angle bracket arrays created in their places would themselves contain subscripts, for example,

$$\text{‘}3\ \langle k\ \# \rangle_n\ 3\text{’} = \text{‘}3\ \langle k\text{-}1\ \# \rangle_n\ 3\ [k\ \#]_n\ 3\ \langle k\text{-}1\ \# \rangle_n\ 3\ [k\ \#]_n\ 3\ \langle k\text{-}1\ \# \rangle_n\ 3\text{’} \qquad (k \geq 1),$$
$$\text{‘}a\ \langle 0\ [m\ \#^*]\ k\ \# \rangle_n\ b\text{’} = \text{‘}a\ \langle b\ \langle m\text{-}1\ \#^* \rangle\ b\ [m\ \#^*]\ k\text{-}1\ \# \rangle_n\ b\text{’} \qquad (k \geq 2),$$

where #* does not begin with a 2- or higher order hyperseparator when m = 1. For the purposes of Rules A2 and A5 (initial part), a ‘2- or higher order hyperseparator’ includes all (n, 2)-hyperseparators, where $n \geq 1$.

The only other modification to the Angle Bracket Rules is that the backslash in Rule A5b can be either a single backslash ($\backslash_j$, where $j \geq 2$) or double backslash ($\backslash\backslash_m$, for some $1 \leq m < j$, in which case the $\backslash_j$ in the $R_{n,j-1}$ equation would be replaced by $\backslash\backslash_m$).

If the separator $[A_{i,j}(p_{i,j})] = [1\ [B]\ d\ \#_S]_j$, where [B] is a (1, 2)-hyperseparator, Rules A5c-d would apply unless $[A_{i,j}(p_{i,j})] = [1 \backslash\backslash 2]_j = \backslash_j$ (which is already covered by Rules A5a-b).

If $[H\ [S]\ d\ \#^*]_m$ is of the form

$$[H^*]_m = [1\ [S_{1,1}]\ 1\ [S_{1,2}]\ \ldots\ 1\ [S_{1,p(1)}]\ d_1\ \#^*_1]_m,$$

where

$$[S_{1,p(1)}] = \backslash\backslash \qquad (h = 1),$$
$$[S_{i,p(i)}] = [1\ [S_{i+1,1}]\ 1\ [S_{i+1,2}]\ \ldots\ 1\ [S_{i+1,p(i+1)}]\ d_{i+1}\ \#^*_{i+1}] \qquad (1 \leq i < h\text{-}1),$$
$$[S_{h-1,p(h-1)}] = [1\ [S_{h,1}]\ 1\ [S_{h,2}]\ \ldots\ 1\ [S_{h,p(h)-1}]\ 1\ \backslash\backslash_h\ d_h\ \#^*_h] \qquad (h \geq 2),$$

$p(i) \geq 1$, $d_i \geq 2$ and each of $[S_{i,j}]$ is an (i, 2)-hyperseparator, the string replacement equation on page 4 would become

$$\text{‘}a\ \langle\ 0\ [A_1]\ 1\ [A_2]\ \ldots\ 1\ [A_k]\ 1\ [B]\ c\ \#\ \rangle\ b\text{’}$$
$$= \text{‘}a\ \langle\ b\ \langle A_1' \rangle\ b\ [A_1]\ b\ \langle A_2' \rangle\ b\ [A_2]\ \ldots\ b\ \langle A_k' \rangle\ b\ [A_k]\ b\ \langle R_{b,1} \rangle\ b\ [B]\ c\text{-}1\ \#\ \rangle\ b\text{’},$$

where

$$R_{n,i} = \text{‘}b\ \langle A_{i+1,1}' \rangle\ b\ [A_{i+1,1}]\ b\ \langle A_{i+1,2}' \rangle\ b\ [A_{i+1,2}]\ \ldots\ b\ \langle A_{i+1,k(i+1)}' \rangle\ b\ [A_{i+1,k(i+1)}]\ b\ \langle R_{n,i+1} \rangle\ b\ [B_{i+1}]\ c_{i+1}\text{-}1\ \#_{i+1}\text{’} \qquad (1 \leq i < m\text{-}1),$$
$$R_{n,m-1} = \text{‘}b\ \langle A_{m,1}' \rangle\ b\ [A_{m,1}]\ b\ \langle A_{m,2}' \rangle\ b\ [A_{m,2}]\ \ldots\ b\ \langle A_{m,k(m)}' \rangle\ b\ [A_{m,k(m)}]\ b\ \langle R_{n,m} \rangle_m\ b\ [H^*]_m\ c_m\text{-}1\ \#_m\text{’} \qquad (m \geq 2),$$
$$R_{n,m+i-1} = \text{‘}b\ \langle S_{i,1}' \rangle\ b\ [S_{i,1}]\ b\ \langle S_{i,2}' \rangle\ b\ [S_{i,2}]\ \ldots\ b\ \langle S_{i,p(i)-1}' \rangle\ b\ [S_{i,p(i)-1}]\ b\ \langle R_{n,m+i} \rangle\ b\ [S_{i,p(i)}]\ d_i\text{-}1\ \#^*_i\text{’} \qquad (1 \leq i < h),$$
$$R_{n,m+h-1} = \text{‘}b\ \langle S_{h,1}' \rangle\ b\ [S_{h,1}]\ b\ \langle S_{h,2}' \rangle\ b\ [S_{h,2}]\ \ldots\ b\ \langle S_{h,p(h)-1}' \rangle\ b\ [S_{h,p(h)-1}]$$
$$b\ \langle A_1' \rangle\ b\ [A_1]\ b\ \langle A_2' \rangle\ b\ [A_2]\ \ldots\ b\ \langle A_k' \rangle\ b\ [A_k]\ b\ \langle R_{n-1,1} \rangle\ b\ [B]\ c\text{-}1\ \#\ \ \backslash\backslash_h\ \ d_h\text{-}1\ \#^*_h\text{’},$$
$$R_{1,1} = \text{‘}0\text{’}$$

and $S_{i,j}'$ is identical to $S_{i,j}$ except that the first entry is reduced by 1. (If m = 1, then [B] = [H*].) This is similar to Rules A5d and A5b – the former executed m+h-1 times (to find equations for initial string

$S_{1,1}$ and $R_{b,1}$ to $R_{b,m+h-2}$), followed by the latter rule ($R_{b,m+h-1}$ onwards) – except that the single backslash is a double backslash.

If the $\backslash\backslash_h$ symbol within the $[S_{h-1,p(h-1)}]$ separator (or the $[S_{1,p(1)}]$ separator within $[H^*]_m$) is replaced by a string of 1's and normal separators (0-hyperseparators) or 1-hyperseparators (below $\theta(\varepsilon_{\Omega+1})$ level), Rules A5a-e would be utilised as for separators below $\theta(\varepsilon_{\Omega+1})$ level in Beyond Bird's Nested Arrays III.

Among the simplest examples of arrays containing a double backslash is

$$N_1 = \{3, 3\ [1\ [1 \backslash\backslash 3]\ 2]\ 2\}.$$

Here, m = h = 1 and

$$[B] = [H^*] = [1 \backslash\backslash 3],$$

and it follows that

$$\begin{aligned}
N_1 &= \{3 \langle 0\ [1 \backslash\backslash 3]\ 2\rangle\ 3\}\\
&= \{3 \langle 3 \langle R_{3,1}\rangle\ 3\rangle\ 3\}\\
&= \{3 \langle 3 \langle 3 \langle R_{2,1}\rangle\ 3 \backslash\backslash 2\rangle\ 3\rangle\ 3\}\\
&= \{3 \langle 3 \langle 3 \langle 3 \langle R_{1,1}\rangle\ 3 \backslash\backslash 2\rangle\ 3 \backslash\backslash 2\rangle\ 3\rangle\ 3\}\\
&= \{3 \langle 3 \langle 3 \langle 3 \backslash\backslash 2\rangle\ 3 \backslash\backslash 2\rangle\ 3\rangle\ 3\}\\
&= \{3 \langle 3 \langle\ 3 \langle 2 \backslash\backslash 2\rangle\ 3\ [3 \backslash\backslash 2]\ 3 \langle 2 \backslash\backslash 2\rangle\ 3\ [3 \backslash\backslash 2]\ 3 \langle 2 \backslash\backslash 2\rangle\ 3\ \backslash\backslash\ 2 \rangle\ 3\rangle\ 3\}\\
&= \{3 \langle 3 \langle\ 3 \backslash 3 \backslash 3 \bullet 3 \backslash 3 \backslash 3 \bullet 3 \backslash 3 \backslash 3\ [3 \backslash\backslash 2]\\
&\qquad 3 \backslash 3 \backslash 3 \bullet 3 \backslash 3 \backslash 3 \bullet 3 \backslash 3 \backslash 3\ [3 \backslash\backslash 2]\\
&\qquad 3 \backslash 3 \backslash 3 \bullet 3 \backslash 3 \backslash 3 \bullet 3 \backslash 3 \backslash 3\ \backslash\backslash\ 2 \rangle\ 3\rangle\ 3\},
\end{aligned}$$

using Rules A2 and A6, with $\bullet$ as shorthand for $[2 \backslash\backslash 2]$. Under Rule A2,

$$\text{'}3 \langle 1 \backslash\backslash 2\rangle\ 3\text{'} = \text{'}3 \backslash 3 \backslash 3\text{'}$$

since the double backslash ($\backslash\backslash$) counts as a '2- or higher order hyperseparator' and the $[1 \backslash\backslash 2]$ symbol 'drops down' to a single backslash ($\backslash$).

A slightly more complicated example is

$$N_2 = \{3, 3\ \ [1\ [1\ [1 \backslash\backslash 3]_2\ 3]\ 2]\ \ 2\}.$$

In this case, m = 2, h = 1 and

$$[B] = [1\ [1 \backslash\backslash 3]_2\ 3],$$
$$[H^*]_m = [1 \backslash\backslash 3]_2,$$

and we obtain

$$\begin{aligned}
N_2 &= \{3 \langle 0\ [1\ [1 \backslash\backslash 3]_2\ 3]\ 2\rangle\ 3\}\\
&= \{3 \langle 3 \langle R_{3,1}\rangle\ 3\rangle\ 3\}\\
&= \{3 \langle 3 \langle 3 \langle R_{3,2}\rangle_2\ 3\ [1 \backslash\backslash 3]_2\ 2\rangle\ 3\rangle\ 3\}\\
&= \{3 \langle 3 \langle 3 \langle 3 \langle R_{2,1}\rangle\ 3 \backslash\backslash 2\rangle_2\ 3\ [1 \backslash\backslash 3]_2\ 2\rangle\ 3\rangle\ 3\}\\
&= \{3 \langle 3 \langle 3 \langle 3 \langle 3 \langle R_{2,2}\rangle_2\ 3\ [1 \backslash\backslash 3]_2\ 2\rangle\ 3 \backslash\backslash 2\rangle_2\ 3\ [1 \backslash\backslash 3]_2\ 2\rangle\ 3\rangle\ 3\}\\
&= \{3 \langle 3 \langle 3 \langle 3 \langle 3 \langle 3 \langle R_{1,1}\rangle\ 3 \backslash\backslash 2\rangle_2\ 3\ [1 \backslash\backslash 3]_2\ 2\rangle\ 3 \backslash\backslash 2\rangle_2\ 3\ [1 \backslash\backslash 3]_2\ 2\rangle\ 3\rangle\ 3\}\\
&= \{3 \langle 3 \langle 3 \langle 3 \langle 3 \langle 3 \backslash\backslash 2\rangle_2\ 3\ [1 \backslash\backslash 3]_2\ 2\rangle\ 3 \backslash\backslash 2\rangle_2\ 3\ [1 \backslash\backslash 3]_2\ 2\rangle\ 3\rangle\ 3\}\\
&= \{3 \langle 3 \langle 3 \langle 3 \langle\ 3 \backslash_2 3 \backslash_2 3 \bullet_2 3 \backslash_2 3 \backslash_2 3 \bullet_2 3 \backslash_2 3 \backslash_2 3\ \ [3 \backslash\backslash 2]_2\\
&\qquad\qquad 3 \backslash_2 3 \backslash_2 3 \bullet_2 3 \backslash_2 3 \backslash_2 3 \bullet_2 3 \backslash_2 3 \backslash_2 3\ \ [3 \backslash\backslash 2]_2\\
&\qquad\qquad 3 \backslash_2 3 \backslash_2 3 \bullet_2 3 \backslash_2 3 \backslash_2 3 \bullet_2 3 \backslash_2 3 \backslash_2 3\ \ [1 \backslash\backslash 3]_2\ \ 2 \rangle\ 3 \backslash\backslash 2\rangle_2\ 3\ [1 \backslash\backslash 3]_2\ 2\rangle\ 3\rangle\ 3\},
\end{aligned}$$

using Rules A2 and A6 (modified to handle angle bracket subscripts), with $\bullet_2$ as shorthand for $[2 \backslash\backslash 2]_2$. The $[1 \backslash\backslash 2]_2$ symbol 'drops down' to $\backslash_2$.

Another example is

$$N_3 = \{3, 2\ \ [1\ [1\ [1\ [1 \backslash\backslash_2 3]\ 2]_2\ 3]\ 2]\ \ 2\}.$$

In this case, m = h = 2 and

$$[B] = [1\ [1\ [1 \backslash\backslash_2 3]\ 2]_2\ 3],$$

$[H^*]_m = [1\ [1 \backslash\backslash_2\ 3]\ 2]_2$,

$[S_{1,1}] = [1 \backslash\backslash_2\ 3]$,

and we find that

$N_3 = \{3\ \langle 0\ [1\ [1\ [1 \backslash\backslash_2\ 3]\ 2]_2\ 3]\ 2\rangle\ 2\}$

$= \{3\ \langle 2\ \langle R_{2,1}\rangle\ 2\rangle\ 2\}$

$= \{3\ \langle 2\ \langle 2\ \langle R_{2,2}\rangle_2\ 2\ \ [1\ [1 \backslash\backslash_2\ 3]\ 2]_2\ \ 2\rangle\ 2\rangle\ 2\}$

$= \{3\ \langle 2\ \langle 2\ \langle 2\ \langle R_{2,3}\rangle\ 2\rangle_2\ 2\ \ [1\ [1 \backslash\backslash_2\ 3]\ 2]_2\ \ 2\rangle\ 2\rangle\ 2\}$

$= \{3\ \langle 2\ \langle 2\ \langle 2\ \langle 2\ \langle R_{1,1}\rangle\ 2 \backslash\backslash_2\ 2\rangle\ 2\rangle_2\ 2\ \ [1\ [1 \backslash\backslash_2\ 3]\ 2]_2\ \ 2\rangle\ 2\rangle\ 2\}$

$= \{3\ \langle 2\ \langle 2\ \langle 2\ \langle 2 \backslash\backslash_2\ 2\rangle\ 2\rangle_2\ 2\ \ [1\ [1 \backslash\backslash_2\ 3]\ 2]_2\ \ 2\rangle\ 2\rangle\ 2\}$

$= \{3\ \langle 2\ \langle 2\ \langle\ 2\ \langle 1 \backslash\backslash_2\ 2\rangle\ 2\ [2 \backslash\backslash_2\ 2]\ 2\ \langle 1 \backslash\backslash_2\ 2\rangle\ 2\ \rangle_2\ 2\ \ [1\ [1 \backslash\backslash_2\ 3]\ 2]_2\ \ 2\rangle\ 2\rangle\ 2\}$

$= \{3\ \langle 2\ \langle 2\ \langle\ 2 \backslash\backslash\ 2\ [2 \backslash\backslash_2\ 2]\ 2 \backslash\backslash\ 2\ \rangle_2\ 2\ \ [1\ [1 \backslash\backslash_2\ 3]\ 2]_2\ \ 2\rangle\ 2\rangle\ 2\}$,

using Rules A2 and A6. The $[1 \backslash\backslash_2\ 2]$ symbol 'drops down' to a double backslash (\\). Since

'$2\ \langle\ 2 \backslash\backslash\ 2\ [2 \backslash\backslash_2\ 2]\ 2 \backslash\backslash\ 2\ \rangle_2\ 2$'

$=$ '$2\ \langle\ 1 \backslash\backslash\ 2\ [2 \backslash\backslash_2\ 2]\ 2 \backslash\backslash\ 2\ \rangle_2\ 2\ \ [2 \backslash\backslash\ 2\ [2 \backslash\backslash_2\ 2]\ 2 \backslash\backslash\ 2]_2\ \ 2\ \langle\ 1 \backslash\backslash\ 2\ [2 \backslash\backslash_2\ 2]\ 2 \backslash\backslash\ 2\ \rangle_2\ 2$'

$=$ '$2\ \ [1 \backslash\backslash\ 2\ [2 \backslash\backslash_2\ 2]\ 2 \backslash\backslash\ 2]_2\ \ 2\ \ [2 \backslash\backslash\ 2\ [2 \backslash\backslash_2\ 2]\ 2 \backslash\backslash\ 2]_2\ \ 2\ \ [1 \backslash\backslash\ 2\ [2 \backslash\backslash_2\ 2]\ 2 \backslash\backslash\ 2]_2\ \ 2$',

using Rules A2 and A6 (modified to handle angle bracket subscripts), it follows that

$N_3 = \{3\ \langle 2\ \langle\ 2\ \ [1 \backslash\backslash\ 2\ [2 \backslash\backslash_2\ 2]\ 2 \backslash\backslash\ 2]_2\ \ 2\ \ [2 \backslash\backslash\ 2\ [2 \backslash\backslash_2\ 2]\ 2 \backslash\backslash\ 2]_2$
$\quad 2\ \ [1 \backslash\backslash\ 2\ [2 \backslash\backslash_2\ 2]\ 2 \backslash\backslash\ 2]_2\ \ 2\ \ [1\ [1 \backslash\backslash_2\ 3]\ 2]_2\ \ 2\ \rangle\ 2\rangle\ 2\}$.

Now I will bring in another all-new 1-hyperseparator symbol – the double black circle (●●). The smallest separator containing this symbol is the [1 ●● 2] separator.

The $\theta(\varepsilon(\varepsilon_{\Omega+1}))$ level separator

$\{a, b\ [1 \bullet\bullet\ 2]\ 2\} = \{a\ \langle 0 \bullet\bullet\ 2\rangle\ b\}$

$= \{a\ \langle b\ \langle b\ \langle\ \ldots\ \langle b\ \langle b \backslash\backslash_b\ b\rangle\ b\rangle\ \ldots\ \rangle\ b\rangle\ b\rangle\ b\}$,

(with b+1 pairs of angle brackets).

This is equivalent to

$\{a, b\ \ [1\ [1\ [1\ [\ \ldots\ [1\ [1 \backslash\backslash_b\ 1, 2]\ 2]\ \ldots\ ]\ 2]\ 2]\ 2]\ \ 2\}$ (with b+1 pairs of square brackets).

$[1 \bullet\bullet\ 2]$ has level $\theta(\varepsilon(\varepsilon_{\Omega+1}))$,

$[1\ [1 \bullet\bullet\ 2]\ 2 \bullet\bullet\ 2]$ has level $\theta(\varepsilon(\varepsilon_{\Omega+1}))2$,

$[1 \backslash\ 2 \bullet\bullet\ 2]$ has level $\varepsilon(\theta(\varepsilon(\varepsilon_{\Omega+1}))+1) = \theta(1, \theta(\varepsilon(\varepsilon_{\Omega+1}))+1)$,

$[1\ [1 \backslash_2\ 3]\ 2 \bullet\bullet\ 2]$ has level $\Gamma(\theta(\varepsilon(\varepsilon_{\Omega+1}))+1) = \theta(\Omega, \theta(\varepsilon(\varepsilon_{\Omega+1}))+1)$,

$[1 \bullet\ 2 \bullet\bullet\ 2]$ has level $\theta(\varepsilon_{\Omega+1}, \theta(\varepsilon(\varepsilon_{\Omega+1}))+1)$,

$[1\ [1 \backslash\backslash\ 3]\ 2 \bullet\bullet\ 2]$ has level $\theta(\varepsilon_{\Omega 2}, \theta(\varepsilon(\varepsilon_{\Omega+1}))+1)$,

$[1\ [1 \backslash\backslash\ 1 \backslash\backslash\ 2]\ 2 \bullet\bullet\ 2]$ has level $\theta(\varepsilon_{\Omega\text{\textasciicircum}2}, \theta(\varepsilon(\varepsilon_{\Omega+1}))+1)$,

$[1\ [1\ [1 \backslash\backslash_2\ 3]\ 2]\ 2 \bullet\bullet\ 2]$ has level $\theta(\varepsilon_{\Omega\text{\textasciicircum}\Omega}, \theta(\varepsilon(\varepsilon_{\Omega+1}))+1)$,

$[1 \bullet\bullet\ 3]$ has level $\theta(\varepsilon(\varepsilon_{\Omega+1}), 1)$,

$[1 \bullet\bullet\ 1\ [1 \bullet\bullet\ 2]\ 2]$ has level $\theta(\varepsilon(\varepsilon_{\Omega+1}), \theta(\varepsilon(\varepsilon_{\Omega+1})))$,

$[1 \bullet\bullet\ 1 \backslash\ 2]$ has level $\theta(\varepsilon(\varepsilon_{\Omega+1})+1)$,

$[1 \bullet\bullet\ 1 \bullet\ 2]$ has level $\theta(\varepsilon(\varepsilon_{\Omega+1})+\varepsilon_{\Omega+1})$,

$[1 \bullet\bullet\ 1\ [1 \backslash\backslash\ 3]\ 2]$ has level $\theta(\varepsilon(\varepsilon_{\Omega+1})+\varepsilon_{\Omega 2})$,

$[1 \bullet\bullet\ 1 \bullet\bullet\ 2]$ has level $\theta(\varepsilon(\varepsilon_{\Omega+1})2)$,

$[1\ [2 \bullet\bullet_2\ 2]\ 2]$ has level $\theta(\varepsilon(\varepsilon_{\Omega+1})\omega)$,

$[1\ [1 \bullet\bullet\ 2 \bullet\bullet_2\ 2]\ 2]$ has level $\theta(\varepsilon(\varepsilon_{\Omega+1})\theta(\varepsilon(\varepsilon_{\Omega+1})))$,

$[1\ [1 \backslash_2\ 2 \bullet\bullet_2\ 2]\ 2]$ has level $\theta(\varepsilon(\varepsilon_{\Omega+1})\Omega)$,

$[1\ [1 \backslash_2\ 1 \backslash_2\ 2 \bullet\bullet_2\ 2]\ 2]$ has level $\theta(\varepsilon(\varepsilon_{\Omega+1})(\Omega\text{\textasciicircum}\Omega))$,

$[1\ [1\ [1 \backslash_3\ 3]\ 2 \bullet\bullet_2\ 2]\ 2]$ has level $\theta(\varepsilon(\varepsilon_{\Omega+1})(\Omega\text{\textasciicircum}\Omega\text{\textasciicircum}\Omega))$,

$[1\ [1 \bullet_2\ 2 \bullet\bullet_2\ 2]\ 2]$ has level $\theta(\varepsilon(\varepsilon_{\Omega+1})\varepsilon_{\Omega+1})$,

[1 [1 [3 \\ 2]$_2$ 2 ●●$_2$ 2] 2] has level θ(ε(ε$_{\Omega+1}$)ε$_{\Omega+2}$),
[1 [1 [1 \\ 3]$_2$ 2 ●●$_2$ 2] 2] has level θ(ε(ε$_{\Omega+1}$)ε$_{\Omega2}$),
[1 [1 [1 \\ 1 \\ 2]$_2$ 2 ●●$_2$ 2] 2] has level θ(ε(ε$_{\Omega+1}$)ε$_{\Omega\text{^}2}$),
[1 [1 [1 [1 \\$_2$ 3] 2]$_2$ 2 ●●$_2$ 2] 2] has level θ(ε(ε$_{\Omega+1}$)ε$_{\Omega\text{^}\Omega}$),
[1 [1 [1 [1 \\$_2$ 1 \\$_2$ 2] 2]$_2$ 2 ●●$_2$ 2] 2] has level θ(ε(ε$_{\Omega+1}$)ε$_{\Omega\text{^}\Omega\text{^}\Omega}$),
[1 [1 [1 [1 [1 \\$_3$ 3] 2] 2]$_2$ 2 ●●$_2$ 2] 2] has level θ(ε(ε$_{\Omega+1}$)ε$_{\Omega\text{^}\Omega\text{^}\Omega\text{^}\Omega}$),
[1 [1 ●●$_2$ 3] 2] has level θ(ε(ε$_{\Omega+1}$)^2),
[1 [1 ●●$_2$ 4] 2] has level θ(ε(ε$_{\Omega+1}$)^3),
[1 [1 ●●$_2$ 1 ●● 2] 2] has level θ(ε(ε$_{\Omega+1}$)^θ(ε(ε$_{\Omega+1}$))),
[1 [1 ●●$_2$ 1 \$_2$ 2] 2] has level θ(ε(ε$_{\Omega+1}$)^Ω),
[1 [1 ●●$_2$ 1 ●$_2$ 2] 2] has level θ(ε(ε$_{\Omega+1}$)^ε$_{\Omega+1}$),
[1 [1 ●●$_2$ 1 [3 \\ 2]$_2$ 2] 2] has level θ(ε(ε$_{\Omega+1}$)^ε$_{\Omega+2}$),
[1 [1 ●●$_2$ 1 [1 \\ 3]$_2$ 2] 2] has level θ(ε(ε$_{\Omega+1}$)^ε$_{\Omega2}$),
[1 [1 ●●$_2$ 1 [1 \\ 1 \\ 2]$_2$ 2] 2] has level θ(ε(ε$_{\Omega+1}$)^ε$_{\Omega\text{^}2}$),
[1 [1 ●●$_2$ 1 [1 [1 \\$_2$ 3] 2]$_2$ 2] 2] has level θ(ε(ε$_{\Omega+1}$)^ε$_{\Omega\text{^}\Omega}$),
[1 [1 ●●$_2$ 1 [1 [1 \\$_2$ 1 \\$_2$ 2] 2]$_2$ 2] 2] has level θ(ε(ε$_{\Omega+1}$)^ε$_{\Omega\text{^}\Omega\text{^}\Omega}$),
[1 [1 ●●$_2$ 1 [1 [1 [1 \\$_3$ 3] 2] 2]$_2$ 2] 2] has level θ(ε(ε$_{\Omega+1}$)^ε$_{\Omega\text{^}\Omega\text{^}\Omega\text{^}\Omega}$),
[1 [1 ●●$_2$ 1 ●●$_2$ 2] 2] has level θ(ε(ε$_{\Omega+1}$)^ε(ε$_{\Omega+1}$)),
[1 [1 ●●$_2$ 1 ●●$_2$ 1 ●●$_2$ 2] 2] has level θ(ε(ε$_{\Omega+1}$)^ε(ε$_{\Omega+1}$)^2),
[1 [1 [2 ●●$_3$ 2] 2] 2] has level θ(ε(ε$_{\Omega+1}$)^ε(ε$_{\Omega+1}$)^ω),
[1 [1 [1 ●●$_3$ 3] 2] 2] has level θ(ε(ε$_{\Omega+1}$)^ε(ε$_{\Omega+1}$)^ε(ε$_{\Omega+1}$)),
[1 [1 [1 ●●$_3$ 1 ●●$_3$ 2] 2] 2] has level θ(ε(ε$_{\Omega+1}$)^ε(ε$_{\Omega+1}$)^ε(ε$_{\Omega+1}$)^ε(ε$_{\Omega+1}$)),
[1 [1 [1 [1 ●●$_4$ 3] 2] 2] 2] has level θ(ε(ε$_{\Omega+1}$)^ε(ε$_{\Omega+1}$)^ε(ε$_{\Omega+1}$)^ε(ε$_{\Omega+1}$)^ε(ε$_{\Omega+1}$)).

The sequence of separators starting with the last three has limit ordinal θ(ε(ε$_{\Omega+1}$+1)).

The next stage launches the treble backslash (\\\), which requires a minimum of three pairs of square brackets around it. The symbol \\$_n$ = [1 \\\ 2]$_n$ in order to mirror \$_n$ = [1 \\ 2]$_n$, and just as \$_n$ = [1 \$_{n+1}$ 2] and \\$_n$ = [1 \\$_{n+1}$ 2], the symbol \\\$_n$ = [1 \\\$_{n+1}$ 2]. The (n, 3)-hyperseparator \\\$_n$ needs a minimum enclosure of n+2 pairs of square brackets. Since I have exhausted \\ = [1 \\\ 2] prior to introducing the ●● symbol, ●●$_n$ = [1 [2 \\\ 2] 2]$_n$. The symbol [2 [2 \\\ 2] 2] comes next in the sequence after ●●.

[1 [2 [2 \\\ 2] 2] 2] has level θ(ε(ε$_{\Omega+1}$+1)),
[1 [3 [2 \\\ 2] 2] 2] has level θ(ε(ε$_{\Omega+1}$+2)),
[1 [1 ●● 2 [2 \\\ 2] 2] 2] has level θ(ε(ε$_{\Omega+1}$+θ(ε(ε$_{\Omega+1}$)))),
[1 [1 \\ 2 [2 \\\ 2] 2] 2] has level θ(ε(ε$_{\Omega+1}$+Ω)),
[1 [1 [1 \\$_2$ 3] 2 [2 \\\ 2] 2] 2] has level θ(ε(ε$_{\Omega+1}$+Ω^Ω)),
[1 [1 [1 \\$_2$ 1 \\$_2$ 2] 2 [2 \\\ 2] 2] 2] has level θ(ε(ε$_{\Omega+1}$+Ω^Ω^Ω)),
[1 [1 [1 [1 \\$_3$ 3] 2] 2 [2 \\\ 2] 2] 2] has level θ(ε(ε$_{\Omega+1}$+Ω^Ω^Ω^Ω)),
[1 [1 [2 \\\ 2] 3] 2] has level θ(ε(ε$_{\Omega+1}$2)),
[1 [1 [2 \\\ 2] 4] 2] has level θ(ε(ε$_{\Omega+1}$3)),
[1 [1 [2 \\\ 2] 1 ●● 2] 2] has level θ(ε(ε$_{\Omega+1}$θ(ε(ε$_{\Omega+1}$)))),
[1 [1 [2 \\\ 2] 1 \\ 2] 2] has level θ(ε(ε$_{\Omega+1}$Ω)),
[1 [1 [2 \\\ 2] 1 [1 \\$_2$ 3] 2] 2] has level θ(ε(ε$_{\Omega+1}$Ω^Ω)),
[1 [1 [2 \\\ 2] 1 [2 \\\ 2] 2] 2] has level θ(ε(ε$_{\Omega+1}$^2)),
[1 [1 [2 [2 \\\ 2]$_2$ 2] 2] 2] has level θ(ε(ε$_{\Omega+1}$^ω)),
[1 [1 [1 ●● 2 [2 \\\ 2]$_2$ 2] 2] 2] has level θ(ε(ε$_{\Omega+1}$^θ(ε(ε$_{\Omega+1}$)))),
[1 [1 [1 \\$_2$ 2 [2 \\\ 2]$_2$ 2] 2] 2] has level θ(ε(ε$_{\Omega+1}$^Ω)),
[1 [1 [1 \\$_2$ 1 \\$_2$ 2 [2 \\\ 2]$_2$ 2] 2] 2] has level θ(ε(ε$_{\Omega+1}$^Ω^Ω)),
[1 [1 [1 [1 \\$_3$ 3] 2 [2 \\\ 2]$_2$ 2] 2] 2] has level θ(ε(ε$_{\Omega+1}$^Ω^Ω^Ω)),

[1 [1 [1 [2 \\\ 2]$_2$ 3] 2] 2] has level θ(ε(ε$_{Ω+1}^{ε}_{Ω+1}$)),
[1 [1 [1 [2 \\\ 2]$_2$ 4] 2] 2] has level θ(ε(ε$_{Ω+1}^{ε}_{Ω+1}$^2)),
[1 [1 [1 [2 \\\ 2]$_2$ 1 ●● 2] 2] 2] has level θ(ε(ε$_{Ω+1}^{ε}_{Ω+1}$^θ(ε(ε$_{Ω+1}$)))),
[1 [1 [1 [2 \\\ 2]$_2$ 1 \\$_2$ 2] 2] 2] has level θ(ε(ε$_{Ω+1}^{ε}_{Ω+1}$^Ω)),
[1 [1 [1 [2 \\\ 2]$_2$ 1 [1 \\$_3$ 3] 2] 2] 2] has level θ(ε(ε$_{Ω+1}^{ε}_{Ω+1}$^Ω^Ω)),
[1 [1 [1 [2 \\\ 2]$_2$ 1 [2 \\\ 2]$_2$ 2] 2] 2] has level θ(ε(ε$_{Ω+1}^{ε}_{Ω+1}^{ε}_{Ω+1}$)),
[1 [1 [1 [1 [2 \\\ 2]$_3$ 3] 2] 2] 2] has level θ(ε(ε$_{Ω+1}^{ε}_{Ω+1}^{ε}_{Ω+1}^{ε}_{Ω+1}$)),
[1 [1 [3 \\\ 2] 2] 2] has level θ(ε(ε$_{Ω+2}$)),
[1 [1 [4 \\\ 2] 2] 2] has level θ(ε(ε$_{Ω+3}$)),
[1 [1 [1 ●● 2 \\\ 2] 2] 2] has level θ(ε(ε(Ω+θ(ε(ε$_{Ω+1}$))))).

[1 [1 [1 \\\ 3] 2] 2] has level θ(ε(ε$_{Ω2}$)),
[1 [1 [1 \\\ 4] 2] 2] has level θ(ε(ε$_{Ω3}$)),
[1 [1 [1 \\\ 1 ●● 2] 2] 2] has level θ(ε(ε(Ωθ(ε(ε$_{Ω+1}$))))),
[1 [1 [1 \\\ 1 \\\ 2] 2] 2] has level θ(ε(ε$_{Ω^2}$)),
[1 [1 [1 \\\ 1 \\\ 1 \\\ 2] 2] 2] has level θ(ε(ε$_{Ω^3}$)),
[1 [1 [1 [2 \\\$_2$ 2] 2] 2] 2] has level θ(ε(ε$_{Ω^ω}$)),
[1 [1 [1 [1 ●● 2 \\\$_2$ 2] 2] 2] 2] has level θ(ε(ε(Ω^θ(ε(ε$_{Ω+1}$))))),
[1 [1 [1 [1 \\\$_2$ 3] 2] 2] 2] has level θ(ε(ε$_{Ω^Ω}$)),
[1 [1 [1 \\\ 2 [1 \\\$_2$ 3] 2] 2] 2] has level θ(ε(ε$_{Ω^Ω+Ω}$)),
[1 [1 [1 [2 \\\$_2$ 2] 2 [1 \\\$_2$ 3] 2] 2] 2] has level θ(ε(ε$_{Ω^Ω+Ω^ω}$)),
[1 [1 [1 [1 \\\$_2$ 3] 3] 2] 2] has level θ(ε(ε$_{(Ω^Ω)2}$)),
[1 [1 [1 [1 \\\$_2$ 3] 1 \\\ 2] 2] 2] has level θ(ε(ε$_{Ω^{(Ω+1)}}$)),
[1 [1 [1 [1 \\\$_2$ 3] 1 [2 \\\$_2$ 2] 2] 2] 2] has level θ(ε(ε$_{Ω^{(Ω+ω)}}$)),
[1 [1 [1 [1 \\\$_2$ 3] 1 [1 \\\$_2$ 3] 2] 2] 2] has level θ(ε(ε$_{Ω^{(Ω2)}}$)),
[1 [1 [1 [2 \\\$_2$ 3] 2] 2] 2] has level θ(ε(ε$_{Ω^{(Ωω)}}$)),
[1 [1 [1 [1 [1 [1 [1 \\\$_2$ 3] 2] 2] 2 \\\$_2$ 3] 2] 2] 2] has level θ(ε(ε(Ω^(Ωθ(ε(ε$_{Ω^Ω}$)))))),
[1 [1 [1 [1 \\\$_2$ 4] 2] 2] 2] has level θ(ε(ε$_{Ω^{Ω^2}}$)),
[1 [1 [1 [1 \\\$_2$ 5] 2] 2] 2] has level θ(ε(ε$_{Ω^{Ω^3}}$)),
[1 [1 [1 [1 \\\$_2$ 1 [1 [1 [1 \\\$_2$ 3] 2] 2] 2] 2] 2] 2] has level θ(ε(ε(Ω^Ω^θ(ε(ε$_{Ω^Ω}$))))),
[1 [1 [1 [1 \\\$_2$ 1 \\\$_2$ 2] 2] 2] 2] has level θ(ε(ε$_{Ω^{Ω^Ω}}$)),
[1 [1 [1 [1 \\\$_2$ 1 \\\$_2$ 1 \\\$_2$ 2] 2] 2] 2] has level θ(ε(ε$_{Ω^{Ω^{Ω^2}}}$)),
[1 [1 [1 [1 [2 \\\$_3$ 2] 2] 2] 2] 2] has level θ(ε(ε$_{Ω^{Ω^{Ω^ω}}}$)),
[1 [1 [1 [1 [1 [1 [1 [1 \\\$_2$ 1 \\\$_2$ 2] 2] 2] 2] 2 \\\$_3$ 2] 2] 2] 2] 2] has level θ(ε(ε(Ω^Ω^Ω^θ(ε(ε$_{Ω^{Ω^Ω}}$))))),
[1 [1 [1 [1 [1 \\\$_3$ 3] 2] 2] 2] 2] has level θ(ε(ε$_{Ω^{Ω^{Ω^Ω}}}$)),
[1 [1 [1 [1 [1 \\\$_3$ 1 \\\$_3$ 2] 2] 2] 2] 2] has level θ(ε(ε$_{Ω^{Ω^{Ω^{Ω^Ω}}}}$)),
[1 [1 [1 [1 [1 [1 \\\$_4$ 3] 2] 2] 2] 2] 2] has level θ(ε(ε$_{Ω^{Ω^{Ω^{Ω^{Ω^Ω}}}}}$)).

The sequence of separators starting with the last three has limit ordinal θ(ε(ε(ε$_{Ω+1}$))).

At this stage, it is better to rewrite \\$_n$ as \$_{n,2}$ and \\\$_n$ as \$_{n,3}$. The symbols \\ and \\\ can be rewritten as \$_{1,2}$ and \$_{1,3}$ respectively; \$_{n,1}$ is \$_n$ (remove trailing 1's). There are two directions of travel for the generalised (m, n)-hyperseparator double subscript backslash symbol (\$_{m,n}$), since

\$_{m,n}$ = [1 \$_{m+1,n}$ 2] = [1 \$_{1,n+1}$ 2]$_m$ (both [1 \$_{m+1,n}$ 2] and [1 \$_{1,n+1}$ 2]$_m$ 'drop down' to \$_{m,n}$).

\$_{m,n}$ requires a minimum of m+n-1 pairs of square brackets around it in order to be used in an array.

With k pairs of square brackets (k ≥ 2),

{a, b [1 [1 [... [1 [2 \$_{1,k}$ 2] 2] ...] 2] 2] 2} = {a ‹0 [1 [... [1 [2 \$_{1,k}$ 2] 2] ...] 2] 2› b}
= {a ‹b ‹b ‹ ... ‹b ‹b \$_{b,k-1}$ b› b› ... › b› b› b}
(with b+k-2 pairs of angle brackets).

The most significant separators introduced so far (beyond Bachmann-Howard ordinal level) are:-

$[1 [2 \backslash_{1,2} 2] 2]$ has level $\theta(\varepsilon_{\Omega+1})$,
$[1 [2 \backslash_{1,2} 2] 3]$ has level $\theta(\varepsilon_{\Omega+1}, 1)$,
$[1 [2 \backslash_{1,2} 2] 1 \backslash 2]$ has level $\theta(\varepsilon_{\Omega+1}+1)$,
$[1 [2 \backslash_{1,2} 2] 1 [2 \backslash_{1,2} 2] 2]$ has level $\theta(\varepsilon_{\Omega+1}2)$,
$[1 [2 [2 \backslash_{1,2} 2]_2 2] 2]$ has level $\theta(\varepsilon_{\Omega+1}\omega)$,
$[1 [1 [2 \backslash_{1,2} 2] 2 [2 \backslash_{1,2} 2]_2 2] 2]$ has level $\theta(\varepsilon_{\Omega+1}\theta(\varepsilon_{\Omega+1}))$,
$[1 [1 \backslash_2 2 [2 \backslash_{1,2} 2]_2 2] 2]$ has level $\theta(\varepsilon_{\Omega+1}\Omega)$,
$[1 [1 [2 \backslash_{1,2} 2]_2 3] 2]$ has level $\theta(\varepsilon_{\Omega+1}{}^\wedge 2)$,
$[1 [1 [2 \backslash_{1,2} 2]_2 1 \backslash 2] 2]$ has level $\theta(\varepsilon_{\Omega+1}{}^\wedge\varepsilon_0)$,
$[1 [1 [2 \backslash_{1,2} 2]_2 1 [2 \backslash_{1,2} 2] 2] 2]$ has level $\theta(\varepsilon_{\Omega+1}{}^\wedge\theta(\varepsilon_{\Omega+1}))$,
$[1 [1 [2 \backslash_{1,2} 2]_2 1 \backslash_2 2] 2]$ has level $\theta(\varepsilon_{\Omega+1}{}^\wedge\Omega)$,
$[1 [1 [2 \backslash_{1,2} 2]_2 1 [2 \backslash_{1,2} 2]_2 2] 2]$ has level $\theta(\varepsilon_{\Omega+1}{}^\wedge\varepsilon_{\Omega+1})$,
$[1 [1 [1 [2 \backslash_{1,2} 2]_3 3] 2] 2]$ has level $\theta(\varepsilon_{\Omega+1}{}^\wedge\varepsilon_{\Omega+1}{}^\wedge\varepsilon_{\Omega+1})$,
$[1 [3 \backslash_{1,2} 2] 2]$ has level $\theta(\varepsilon_{\Omega+2})$,
$[1 [1 \backslash 2 \backslash_{1,2} 2] 2]$ has level $\theta(\varepsilon(\Omega+\varepsilon_0))$,
$[1 [1 [2 \backslash_{1,2} 2] 2 \backslash_{1,2} 2] 2]$ has level $\theta(\varepsilon(\Omega+\theta(\varepsilon_{\Omega+1})))$,
$[1 [1 \backslash_{1,2} 3] 2]$ has level $\theta(\varepsilon_{\Omega2})$,
$[1 [1 \backslash_{1,2} 1 \backslash_{1,2} 2] 2]$ has level $\theta(\varepsilon_{\Omega{}^\wedge 2})$,
$[1 [1 [2 \backslash_{2,2} 2] 2] 2]$ has level $\theta(\varepsilon_{\Omega{}^\wedge\omega})$,
$[1 [1 [1 \backslash_{2,2} 3] 2] 2]$ has level $\theta(\varepsilon_{\Omega{}^\wedge\Omega})$,
$[1 [1 [1 \backslash_{2,2} 1 \backslash_{2,2} 2] 2] 2]$ has level $\theta(\varepsilon_{\Omega{}^\wedge\Omega{}^\wedge\Omega})$,
$[1 [1 [1 [1 \backslash_{3,2} 3] 2] 2] 2]$ has level $\theta(\varepsilon_{\Omega{}^\wedge\Omega{}^\wedge\Omega{}^\wedge\Omega})$,
$[1 [1 [2 \backslash_{1,3} 2] 2] 2]$ has level $\theta(\varepsilon(\varepsilon_{\Omega+1}))$,
$[1 [2 [2 \backslash_{1,3} 2] 2] 2]$ has level $\theta(\varepsilon(\varepsilon_{\Omega+1}+1))$,
$[1 [1 \backslash_{1,2} 2 [2 \backslash_{1,3} 2] 2] 2]$ has level $\theta(\varepsilon(\varepsilon_{\Omega+1}+\Omega))$,
$[1 [1 [2 \backslash_{1,3} 2] 3] 2]$ has level $\theta(\varepsilon(\varepsilon_{\Omega+1}2))$,
$[1 [1 [2 \backslash_{1,3} 2] 1 \backslash_{1,2} 2] 2]$ has level $\theta(\varepsilon(\varepsilon_{\Omega+1}\Omega))$,
$[1 [1 [2 \backslash_{1,3} 2] 1 [2 \backslash_{1,3} 2] 2] 2]$ has level $\theta(\varepsilon(\varepsilon_{\Omega+1}{}^\wedge 2))$,
$[1 [1 [1 [2 \backslash_{1,3} 2]_2 3] 2] 2]$ has level $\theta(\varepsilon(\varepsilon_{\Omega+1}{}^\wedge\varepsilon_{\Omega+1}))$,
$[1 [1 [3 \backslash_{1,3} 2] 2] 2]$ has level $\theta(\varepsilon(\varepsilon_{\Omega+2}))$,
$[1 [1 [1 \backslash_{1,3} 3] 2] 2]$ has level $\theta(\varepsilon(\varepsilon_{\Omega2}))$,
$[1 [1 [1 \backslash_{1,3} 1 \backslash_{1,3} 2] 2] 2]$ has level $\theta(\varepsilon(\varepsilon_{\Omega{}^\wedge 2}))$,
$[1 [1 [1 [2 \backslash_{2,3} 2] 2] 2] 2]$ has level $\theta(\varepsilon(\varepsilon_{\Omega{}^\wedge\omega}))$,
$[1 [1 [1 [1 \backslash_{2,3} 3] 2] 2] 2]$ has level $\theta(\varepsilon(\varepsilon_{\Omega{}^\wedge\Omega}))$,
$[1 [1 [1 [1 \backslash_{2,3} 1 \backslash_{2,3} 2] 2] 2] 2]$ has level $\theta(\varepsilon(\varepsilon_{\Omega{}^\wedge\Omega{}^\wedge\Omega}))$,
$[1 [1 [1 [1 [1 \backslash_{3,3} 3] 2] 2] 2] 2]$ has level $\theta(\varepsilon(\varepsilon_{\Omega{}^\wedge\Omega{}^\wedge\Omega{}^\wedge\Omega}))$.

Continuing this sequence, I obtain

$[1 [1 \ [1 [2 \backslash_{1,4} 2] 2] \ 2] 2]$ has level $\theta(\varepsilon(\varepsilon(\varepsilon_{\Omega+1})))$,
$[1 [2 \ [1 [2 \backslash_{1,4} 2] 2] \ 2] 2]$ has level $\theta(\varepsilon(\varepsilon(\varepsilon_{\Omega+1})+1))$,
$[1 [1 \ [1 [2 \backslash_{1,4} 2] 2] \ 3] 2]$ has level $\theta(\varepsilon(\varepsilon(\varepsilon_{\Omega+1})2))$,
$[1 [1 \ [1 [2 \backslash_{1,4} 2] 2] \ 1 \ [1 [2 \backslash_{1,4} 2] 2] \ 2] 2]$ has level $\theta(\varepsilon(\varepsilon(\varepsilon_{\Omega+1}){}^\wedge 2))$,
$[1 [1 [2 \ [1 [2 \backslash_{1,4} 2] 2]_2 \ 2] 2] 2]$ has level $\theta(\varepsilon(\varepsilon(\varepsilon_{\Omega+1}){}^\wedge\omega))$,
$[1 [1 [1 \ [1 [2 \backslash_{1,4} 2] 2]_2 \ 3] 2] 2]$ has level $\theta(\varepsilon(\varepsilon(\varepsilon_{\Omega+1}){}^\wedge\varepsilon(\varepsilon_{\Omega+1})))$,
$[1 [1 [2 \ [2 \backslash_{1,4} 2] \ 2] 2] 2]$ has level $\theta(\varepsilon(\varepsilon(\varepsilon_{\Omega+1}+1)))$,
$[1 [1 [1 \ [2 \backslash_{1,4} 2] \ 3] 2] 2]$ has level $\theta(\varepsilon(\varepsilon(\varepsilon_{\Omega+1}2)))$,
$[1 [1 [1 \ [2 \backslash_{1,4} 2] \ 1 \ [2 \backslash_{1,4} 2] \ 2] 2] 2]$ has level $\theta(\varepsilon(\varepsilon(\varepsilon_{\Omega+1}{}^\wedge 2)))$,
$[1 [1 [1 [2 \ [2 \backslash_{1,4} 2]_2 \ 2] 2] 2] 2]$ has level $\theta(\varepsilon(\varepsilon(\varepsilon_{\Omega+1}{}^\wedge\omega)))$,

[1 [1 [1 [1 [2 $\backslash_{1,4}$ 2]$_2$ 3] 2] 2] 2] has level $\theta(\varepsilon(\varepsilon(\varepsilon_{\Omega+1}{}^\wedge\varepsilon_{\Omega+1})))$,
[1 [1 [1 [3 $\backslash_{1,4}$ 2] 2] 2] 2] has level $\theta(\varepsilon(\varepsilon(\varepsilon_{\Omega+2})))$,
[1 [1 [1 [1 $\backslash_{1,4}$ 3] 2] 2] 2] has level $\theta(\varepsilon(\varepsilon(\varepsilon_{\Omega 2})))$,
[1 [1 [1 [1 $\backslash_{1,4}$ 1 $\backslash_{1,4}$ 2] 2] 2] 2] has level $\theta(\varepsilon(\varepsilon(\varepsilon_{\Omega{}^\wedge 2})))$,
[1 [1 [1 [1 [2 $\backslash_{2,4}$ 2] 2] 2] 2] 2] has level $\theta(\varepsilon(\varepsilon(\varepsilon_{\Omega{}^\wedge\omega})))$,
[1 [1 [1 [1 [1 $\backslash_{2,4}$ 3] 2] 2] 2] 2] has level $\theta(\varepsilon(\varepsilon(\varepsilon_{\Omega{}^\wedge\Omega})))$,
[1 [1 [1 [1 [1 $\backslash_{2,4}$ 1 $\backslash_{2,4}$ 2] 2] 2] 2] 2] has level $\theta(\varepsilon(\varepsilon(\varepsilon_{\Omega{}^\wedge\Omega{}^\wedge\Omega})))$,
[1 [1 [1 [1 [1 [1 $\backslash_{3,4}$ 3] 2] 2] 2] 2] 2] has level $\theta(\varepsilon(\varepsilon(\varepsilon_{\Omega{}^\wedge\Omega{}^\wedge\Omega{}^\wedge\Omega})))$,
[1 [1 [1 [1 [2 $\backslash_{1,5}$ 2] 2] 2] 2] 2] has level $\theta(\varepsilon(\varepsilon(\varepsilon(\varepsilon_{\Omega+1}))))$,
[1 [1 [1 [1 [3 $\backslash_{1,5}$ 2] 2] 2] 2] 2] has level $\theta(\varepsilon(\varepsilon(\varepsilon(\varepsilon_{\Omega+2}))))$,
[1 [1 [1 [1 [1 $\backslash_{1,5}$ 3] 2] 2] 2] 2] has level $\theta(\varepsilon(\varepsilon(\varepsilon(\varepsilon_{\Omega 2}))))$,
[1 [1 [1 [1 [1 $\backslash_{1,5}$ 1 $\backslash_{1,5}$ 2] 2] 2] 2] 2] has level $\theta(\varepsilon(\varepsilon(\varepsilon(\varepsilon_{\Omega{}^\wedge 2}))))$,
[1 [1 [1 [1 [1 [1 $\backslash_{2,5}$ 3] 2] 2] 2] 2] 2] has level $\theta(\varepsilon(\varepsilon(\varepsilon(\varepsilon_{\Omega{}^\wedge\Omega}))))$,
[1 [1 [1 [1 [1 [1 $\backslash_{2,5}$ 1 $\backslash_{2,5}$ 2] 2] 2] 2] 2] 2] has level $\theta(\varepsilon(\varepsilon(\varepsilon(\varepsilon_{\Omega{}^\wedge\Omega{}^\wedge\Omega}))))$,
[1 [1 [1 [1 [1 [1 [1 $\backslash_{3,5}$ 3] 2] 2] 2] 2] 2] 2] has level $\theta(\varepsilon(\varepsilon(\varepsilon(\varepsilon_{\Omega{}^\wedge\Omega{}^\wedge\Omega{}^\wedge\Omega}))))$,
[1 [1 [1 [1 [1 [2 $\backslash_{1,6}$ 2] 2] 2] 2] 2] 2] has level $\theta(\varepsilon(\varepsilon(\varepsilon(\varepsilon(\varepsilon_{\Omega+1})))))$,
[1 [1 [1 [1 [1 [1 [2 $\backslash_{1,7}$ 2] 2] 2] 2] 2] 2] 2] has level $\theta(\varepsilon(\varepsilon(\varepsilon(\varepsilon(\varepsilon(\varepsilon_{\Omega+1}))))))$,
[1 [1 [1 [1 [1 [1 [1 [2 $\backslash_{1,8}$ 2] 2] 2] 2] 2] 2] 2] 2] has level $\theta(\varepsilon(\varepsilon(\varepsilon(\varepsilon(\varepsilon(\varepsilon(\varepsilon_{\Omega+1})))))))$.

The sequence of separators starting with the last three has limit ordinal $\theta(\zeta_{\Omega+1}) = \theta(\varphi(2, \Omega+1))$.

If $[H^*]_m$ on page 7 consists of many layers of separators, where

$[X_{i,p(i)}] = [1\ [X_{i+1,1}]\ 1\ [X_{i+1,2}]\ ...\ 1\ [X_{i+1,p(i+1)}]\ k_{i+1}\ \#_{i+1}]$,

eventually finishing up with $[X_{x,p(x)}] = \backslash_{r,s}$ for some r, s and x (instead of $\backslash\backslash_h$), the road from $R_{n,1}$ to $R_{n-1,1}$ would be split up into s parts. The first part would process the i-hyperseparators (each ending in $[B_i]$'s, up to $[B_m] = [H^*]_m$; i increases from 2 to m), the second part would process the (i, 2)-hyperseparators (up to the set ending in, say, $[H^*_2]_{m(2)}$; i increases from 1 to m(2)), the third part would process the (i, 3)-hyperseparators (up to the set ending in $[H^*_3]_{m(3)}$), and so on, up to the final (sth) part, which finishes with the (r, s)-hyperseparators and the 1-hyperseparators all in one string. There would be t = m+m(2)+m(3)+...+m(s)-1 layers in all (note that m(s) = r), with up to s-1 of those layers having subscripts to the square brackets. The $R_{n,t}$ string would contain the (r, s)-hyperseparators and the 1-hyperseparators with the latter set sandwiched in between the penultimate (r, s)-hyperseparator and the $\backslash_{r,s}$ symbol. Rule A5d would be applied t times (to find equations for initial string $S_{1,1}$ and $R_{b,1}$ to $R_{b,t-1}$), followed by Rule A5b ($R_{b,t}$ onwards).

An (m, n)-hyperseparator is either a $\backslash_{m,n}$ symbol or contains at least one $\backslash_{m+k,n}$ symbol inside k pairs of square brackets or at least one $\backslash_{1,n+1}$ symbol or (1, n+1)-hyperseparator inside k+1 pairs of square brackets (with the highest of the layers having an m+k subscript), for some value of k – but at every value of k there are no $\backslash_{m+k,n}$ symbols inside fewer than k pairs of square brackets or $\backslash_{1,n+1}$ symbols or (1, n+1)-hyperseparators inside fewer than k+1 pairs of square brackets (with the highest of the layers having an m+k subscript). An (n, 1)-hyperseparator is an n-hyperseparator.

Putting it another way, a $\backslash_{m,n}$ symbol is an (m, n)-hyperseparator, $\backslash_{m,n}$ enclosed by k pairs of square brackets is an (m-k, n)-hyperseparator (m > k), $\backslash_{m,n}$ enclosed by m pairs of square brackets (subscripted by k at the bottom) is a (k, n-1)-hyperseparator (n ≥ 2), $\backslash_{m,n}$ enclosed by m pairs of square brackets (with no subscript at the bottom) is a (1, n-1)-hyperseparator (n ≥ 2). A $\backslash_{m,n}$ symbol enclosed by m pairs of square brackets (subscripted by m_2 at the bottom), enclosed by m_2 pairs of square brackets (subscripted by m_3 at the bottom), ... , enclosed by m_{n-1} pairs of square brackets (subscripted by m_n at the bottom), enclosed by m_n pairs of square brackets is a normal separator (0-hyperseparator).

The recursive definition of an (m, n)-hyperseparator is that it is either a $\backslash_{m,n}$ symbol or contains either at least one (m+1, n)-hyperseparator in its 'base layer' or is subscripted by m and contains at least one (1, n+1)-hyperseparator in its 'base layer'.

An (m_1, n_1)-hyperseparator cannot be on the same 'nested level' as an (m_2, n_2)-hyperseparator unless both hyperseparators are of the same level (both $m_1 = m_2$ and $n_1 = n_2$) or one (or both) separators are either normal or 1-hyperseparators. Suppose that k separators $[X_1]$, $[X_2]$, ... , $[X_k]$ appear on the same 'nested level' somewhere within a giant normal separator [N], as in this example

$[N] = [\# \; [n_1 \; [X_1] \; n_2 \; [X_2] \; ... \; n_k \; [X_k] \; n_{k+1}] \; \#^*]$ (# and #* represent the remainder of N).

If one of the $[X_i]$ is a 2- or higher hyperseparator, say, an (m, n)-hyperseparator, then each of the $[X_i]$ is either a normal separator, a 1-hyperseparator or an (m, n)-hyperseparator.

Rules A5b and A5b* are modified as follows:-

The single or double backslash in Rule A5b is now the generalised double subscript backslash $\backslash_{r,s}$ symbol, where either r ≥ 2 or s ≥ 2. The $\backslash_j$ in the $R_{n,j-1}$ equation is replaced by $\backslash_{r,s}$.

Rule A5b* (separator $[A_{i,j}(p_{i,j})] = [d \; \#_S]_m$, where d ≥ 2 and $\#_S$ contains at least one (1, k)-hyperseparator in its base layer, where k ≥ 2):

$S_{i,j}$ = 'b ‹$A_{i,j}(1)$'› b [$A_{i,j}(1)$] b ‹$A_{i,j}(2)$'› b [$A_{i,j}(2)$] ... b ‹$A_{i,j}(p_{i,j}-1)$'› b [$A_{i,j}(p_{i,j}-1)$] R_b $[d \; \#_S]_m$ $c_{i,j}$-1 $\#_{i,j}$',

R_n = 'b ‹R_{n-1}› b',

R_1 = 'b $[d\text{-}1 \; \#_S]_{m+b-1}$ b'.

Note that Rule A5b* with the lowest (1, k)-hyperseparator within $\#_S$, $[A_{i,j}(p_{i,j})] = [2 \; \backslash_{1,k} \; 2]_m$ would mean that R_1 = 'b $[1 \; \backslash_{1,k} \; 2]_{m+b-1}$ b' = 'b $\backslash_{m+b-1,k-1}$ b'. By setting m = 1, we achieve R_1 = 'b $\backslash_{b,k-1}$ b', which is how the $\backslash_{1,k}$ symbol is reduced to the $\backslash_{b,k-1}$ symbol.

If the separator $[A_{i,j}(p_{i,j})] = [1 \; [B_1] \; 1 \; [B_2] \; ... \; 1 \; [B_q] \; d \; \#_S]_m$, where q ≥ 1, d ≥ 2, each of $[B_i]$ is either a normal separator or 1-hyperseparator, and $\#_S$ contains at least one (1, k)-hyperseparator in its 'base layer', where k ≥ 2, Rule A5e would apply, but with the separator $[A_{i,j}(p_{i,j})]$ and the associated angle bracket array each carrying the subscript m:

$S_{i,j}$ = 'b ‹$A_{i,j}(1)$'› b [$A_{i,j}(1)$] b ‹$A_{i,j}(2)$'› b [$A_{i,j}(2)$] ... b ‹$A_{i,j}(p_{i,j}-1)$'› b [$A_{i,j}(p_{i,j}-1)$]
b ‹$A_{i,j}(p_{i,j})$'›$_m$ b [$A_{i,j}(p_{i,j})$]$_m$ $c_{i,j}$-1 $\#_{i,j}$'.

I can now expand the double subscript in the backslash symbol so that it too becomes an array in its own right. The single backslash with no subscripts has an infinite number of pathways, since

$\backslash = [1 \; \backslash_2 \; 2] = [1 \; \backslash_{1,2} \; 2] = [1 \; \backslash_{1,1,2} \; 2] = [1 \; \backslash_{1,1,1,2} \; 2] = ...$

When there are k subscripts, the $(n_1, n_2, n_3, ... , n_k)$-hyperseparator backslash symbol

$$\begin{aligned} \backslash_{n_1,n_2,n_3,...,n_k} &= [1 \; \backslash_{n_1+1,n_2,n_3,...,n_k} \; 2] \\ &= [1 \; \backslash_{1,n_2+1,n_3,...,n_k} \; 2]_{n_1} \\ &= [1 \; \backslash_{1,1,n_3+1,n_4,...,n_k} \; 2]_{n_1,n_2} \\ &= ...... \\ &= [1 \; \backslash_{1,1,1,...,1,n_k+1} \; 2]_{n_1,n_2,n_3,...,n_{k-1}} && \text{(with k-1 1's)} \\ &= [1 \; \backslash_{1,1,1,...,1,2} \; 2]_{n_1,n_2,n_3,...,n_k} && \text{(with k or more 1's)} \end{aligned}$$

requires a minimum of $n_1+n_2+n_3+...+n_k-(k-1)$ pairs of square brackets around it in order to turn in into a normal separator and be used in the 'base layer' of a curly bracket array. The backslash itself in the above equalities (other than the last one) may be substituted by a separator array that contains at least one (1, 1, 1, ... , 1, r_1, r_2, ...)-hyperseparator (with at least k 1's and r_1 ≥ 2) in its 'base layer', for example,

$[X \backslash_{1,1,1,...,1,m} Y]_{n_1,n_2,n_3,...,n_k} = [1 [X \backslash_{1,1,1,...,1,m} Y]_{1,1,n_3+1,n_4,...,n_k} 2]_{n_1,n_2}$
(with k 1's prior to m ≥ 2; X and Y are strings either side of $\backslash_{1,1,1,...,1,m}$).

The $\theta(\zeta_{\Omega+1})$ level separator

$\{a, b [1 [2 \backslash_{1,1,2} 2] 2] 2\} = \{a \langle 0 [2 \backslash_{1,1,2} 2] 2\rangle b\}$
$= \{a \langle b \langle b \langle ... \langle b \langle b \backslash_{1,b} b\rangle b\rangle ... \rangle b\rangle b\rangle b\}$
(with b pairs of angle brackets).

This is equivalent to

$\{a, b [1 [1 [1 [... [1 [1 \backslash_{1,b} 1, 2] 2] ...] 2] 2] 2] 2\}$ (with b pairs of square brackets).

In general, with k 1's in the subscript,

$\{a, b [1 [c \backslash_{1,1,1,...,1,2} 2] 2] 2\} = \{a \langle 0 [c \backslash_{1,1,1,...,1,2} 2] 2\rangle b\}$
$= \{a \langle b \langle b \langle ... \langle b \langle b [c-1 \backslash_{1,1,1,...,1,2} 2]_{1,1,...,1,b} b\rangle b\rangle ... \rangle b\rangle b\rangle b\}$
(with b pairs of angle brackets and k-1 1's in $_{1,1,...,1,b}$).

When c = 2, the separator $[1 \backslash_{1,1,1,...,1,2} 2]_{1,1,...,1,b}$ 'drops down' to $\backslash_{1,1,...,1,b}$ (with k-1 1's in $_{1,1,...,1,b}$).

$[1 [2 \backslash_{1,1,2} 2] 2]$ has level $\theta(\zeta_{\Omega+1}) = \theta(\varphi(2, \Omega+1))$,
$[1 [2 \backslash_{1,1,2} 2] 3]$ has level $\theta(\zeta_{\Omega+1}, 1)$,
$[1 [2 \backslash_{1,1,2} 2] 1 \backslash 2]$ has level $\theta(\zeta_{\Omega+1}+1)$,
$[1 [2 \backslash_{1,1,2} 2] 1 [2 \backslash_{1,1,2} 2] 2]$ has level $\theta(\zeta_{\Omega+1}2)$,
$[1 [2 [2 \backslash_{1,1,2} 2]_2 2] 2]$ has level $\theta(\zeta_{\Omega+1}\omega)$,
$[1 [1 \backslash 2 [2 \backslash_{1,1,2} 2]_2 2] 2]$ has level $\theta(\zeta_{\Omega+1}\varepsilon_0)$,
$[1 [1 [2 \backslash_{1,1,2} 2] 2 [2 \backslash_{1,1,2} 2]_2 2] 2]$ has level $\theta(\zeta_{\Omega+1}\theta(\zeta_{\Omega+1}))$,
$[1 [1 \backslash_2 2 [2 \backslash_{1,1,2} 2]_2 2] 2]$ has level $\theta(\zeta_{\Omega+1}\Omega)$,
$[1 [1 \bullet_2 2 [2 \backslash_{1,1,2} 2]_2 2] 2]$ has level $\theta(\zeta_{\Omega+1}\varepsilon_{\Omega+1})$ ($\bullet_2 = [2 \backslash_{1,2} 2]_2$),
$[1 [1 \bullet\bullet_2 2 [2 \backslash_{1,1,2} 2]_2 2] 2]$ has level $\theta(\zeta_{\Omega+1}\varepsilon(\varepsilon_{\Omega+1}))$ ($\bullet\bullet_2 = [1 [2 \backslash_{1,3} 2] 2]_2$),
$[1 [1 [2 \backslash_{1,1,2} 2]_2 3] 2]$ has level $\theta(\zeta_{\Omega+1}\text{^}2)$,
$[1 [1 [2 \backslash_{1,1,2} 2]_2 1 \backslash_2 2] 2]$ has level $\theta(\zeta_{\Omega+1}\text{^}\Omega)$,
$[1 [1 [2 \backslash_{1,1,2} 2]_2 1 \bullet_2 2] 2]$ has level $\theta(\zeta_{\Omega+1}\text{^}\varepsilon_{\Omega+1})$,
$[1 [1 [2 \backslash_{1,1,2} 2]_2 1 \bullet\bullet_2 2] 2]$ has level $\theta(\zeta_{\Omega+1}\text{^}\varepsilon(\varepsilon_{\Omega+1}))$,
$[1 [1 [2 \backslash_{1,1,2} 2]_2 1 [2 \backslash_{1,1,2} 2]_2 2] 2]$ has level $\theta(\zeta_{\Omega+1}\text{^}\zeta_{\Omega+1})$,
$[1 [1 [1 [2 \backslash_{1,1,2} 2]_3 3] 2] 2]$ has level $\theta(\zeta_{\Omega+1}\text{^}\zeta_{\Omega+1}\text{^}\zeta_{\Omega+1})$,
$[1 [1 [1 [2 \backslash_{1,1,2} 2]_3 1 [2 \backslash_{1,1,2} 2]_3 2] 2] 2]$ has level $\theta(\zeta_{\Omega+1}\text{^}\zeta_{\Omega+1}\text{^}\zeta_{\Omega+1}\text{^}\zeta_{\Omega+1})$,
$[1 [1 [1 [1 [2 \backslash_{1,1,2} 2]_4 3] 2] 2] 2]$ has level $\theta(\zeta_{\Omega+1}\text{^}\zeta_{\Omega+1}\text{^}\zeta_{\Omega+1}\text{^}\zeta_{\Omega+1}\text{^}\zeta_{\Omega+1})$.

$[1 [2 [2 \backslash_{1,1,2} 2]_{1,2} 2] 2]$ has level $\theta(\varepsilon(\zeta_{\Omega+1}+1))$ ($[2 \backslash_{1,1,2} 2] = [1 [2 \backslash_{1,1,2} 2]_{1,2} 2]$),
$[1 [3 [2 \backslash_{1,1,2} 2]_{1,2} 2] 2]$ has level $\theta(\varepsilon(\zeta_{\Omega+1}+2))$,
$[1 [1 \backslash_{1,2} 2 [2 \backslash_{1,1,2} 2]_{1,2} 2] 2]$ has level $\theta(\varepsilon(\zeta_{\Omega+1}+\Omega))$,
$[1 [1 [2 \backslash_{1,3} 2] 2 [2 \backslash_{1,1,2} 2]_{1,2} 2] 2]$ has level $\theta(\varepsilon(\zeta_{\Omega+1}+\varepsilon_{\Omega+1}))$,
$[1 [1 [1 [2 \backslash_{1,4} 2] 2] 2 [2 \backslash_{1,1,2} 2]_{1,2} 2] 2]$ has level $\theta(\varepsilon(\zeta_{\Omega+1}+\varepsilon(\varepsilon_{\Omega+1})))$,
$[1 [1 [2 \backslash_{1,1,2} 2]_{1,2} 3] 2]$ has level $\theta(\varepsilon(\zeta_{\Omega+1}2))$,
$[1 [1 [2 \backslash_{1,1,2} 2]_{1,2} 1 \backslash_{1,2} 2] 2]$ has level $\theta(\varepsilon(\zeta_{\Omega+1}\Omega))$,
$[1 [1 [2 \backslash_{1,1,2} 2]_{1,2} 1 [2 \backslash_{1,3} 2] 2] 2]$ has level $\theta(\varepsilon(\zeta_{\Omega+1}\varepsilon_{\Omega+1}))$,
$[1 [1 [2 \backslash_{1,1,2} 2]_{1,2} 1 [1 [2 \backslash_{1,4} 2] 2] 2] 2]$ has level $\theta(\varepsilon(\zeta_{\Omega+1}\varepsilon(\varepsilon_{\Omega+1})))$,
$[1 [1 [2 \backslash_{1,1,2} 2]_{1,2} 1 [2 \backslash_{1,1,2} 2]_{1,2} 2] 2]$ has level $\theta(\varepsilon(\zeta_{\Omega+1}\text{^}2))$,
$[1 [1 [2 [2 \backslash_{1,1,2} 2]_{2,2} 2] 2] 2]$ has level $\theta(\varepsilon(\zeta_{\Omega+1}\text{^}\omega))$,
$[1 [1 [1 \backslash_{2,2} 2 [2 \backslash_{1,1,2} 2]_{2,2} 2] 2] 2]$ has level $\theta(\varepsilon(\zeta_{\Omega+1}\text{^}\Omega))$,
$[1 [1 [1 [2 \backslash_{1,3} 2]_2 2 [2 \backslash_{1,1,2} 2]_{2,2} 2] 2] 2]$ has level $\theta(\varepsilon(\zeta_{\Omega+1}\text{^}\varepsilon_{\Omega+1}))$ ($\backslash_{2,2} = [1 \backslash_{1,3} 2]_2$),
$[1 [1 [1 [1 [2 \backslash_{1,4} 2] 2]_2 2 [2 \backslash_{1,1,2} 2]_{2,2} 2] 2] 2]$ has level $\theta(\varepsilon(\zeta_{\Omega+1}\text{^}\varepsilon(\varepsilon_{\Omega+1})))$,

$[1\ [1\ [1\ \ [2 \backslash_{1,1,2}\ 2]_{2,2}\ \ 3]\ 2]\ 2]$ has level $\theta(\varepsilon(\zeta_{\Omega+1}\hat{}\zeta_{\Omega+1}))$,
$[1\ [1\ [1\ \ [2 \backslash_{1,1,2}\ 2]_{2,2}\ \ 1 \backslash_{2,2}\ 2]\ 2]\ 2]$ has level $\theta(\varepsilon(\zeta_{\Omega+1}\hat{}\zeta_{\Omega+1}\hat{}\Omega))$,
$[1\ [1\ [1\ \ [2 \backslash_{1,1,2}\ 2]_{2,2}\ \ 1\ [2 \backslash_{1,3}\ 2]_2\ 2]\ 2]\ 2]$ has level $\theta(\varepsilon(\zeta_{\Omega+1}\hat{}\zeta_{\Omega+1}\hat{}\varepsilon_{\Omega+1}))$,
$[1\ [1\ [1\ \ [2 \backslash_{1,1,2}\ 2]_{2,2}\ \ 1\ [1\ [2 \backslash_{1,4}\ 2]\ 2]_2\ 2]\ 2]\ 2]$ has level $\theta(\varepsilon(\zeta_{\Omega+1}\hat{}\zeta_{\Omega+1}\hat{}\varepsilon(\varepsilon_{\Omega+1})))$,
$[1\ [1\ [1\ \ [2 \backslash_{1,1,2}\ 2]_{2,2}\ \ 1\ \ [2 \backslash_{1,1,2}\ 2]_{2,2}\ \ 2]\ 2]\ 2]$ has level $\theta(\varepsilon(\zeta_{\Omega+1}\hat{}\zeta_{\Omega+1}\hat{}\zeta_{\Omega+1}))$,
$[1\ [1\ [1\ [1\ \ [2 \backslash_{1,1,2}\ 2]_{3,2}\ \ 3]\ 2]\ 2]\ 2]$ has level $\theta(\varepsilon(\zeta_{\Omega+1}\hat{}\zeta_{\Omega+1}\hat{}\zeta_{\Omega+1}\hat{}\zeta_{\Omega+1}))$,
$[1\ [1\ [1\ [1\ \ [2 \backslash_{1,1,2}\ 2]_{3,2}\ \ 1\ \ [2 \backslash_{1,1,2}\ 2]_{3,2}\ \ 2]\ 2]\ 2]\ 2]$ has level $\theta(\varepsilon(\zeta_{\Omega+1}\hat{}\zeta_{\Omega+1}\hat{}\zeta_{\Omega+1}\hat{}\zeta_{\Omega+1}\hat{}\zeta_{\Omega+1}))$,
$[1\ [1\ [1\ [1\ [1\ \ [2 \backslash_{1,1,2}\ 2]_{4,2}\ \ 3]\ 2]\ 2]\ 2]\ 2]$ has level $\theta(\varepsilon(\zeta_{\Omega+1}\hat{}\zeta_{\Omega+1}\hat{}\zeta_{\Omega+1}\hat{}\zeta_{\Omega+1}\hat{}\zeta_{\Omega+1}\hat{}\zeta_{\Omega+1}))$.

$[1\ [1\ [2\ \ [2 \backslash_{1,1,2}\ 2]_{1,3}\ \ 2]\ 2]\ 2]$ has level $\theta(\varepsilon(\varepsilon(\zeta_{\Omega+1}+1)))$ $([2 \backslash_{1,1,2}\ 2]_{1,n} = [1\ [2 \backslash_{1,1,2}\ 2]_{1,n+1}\ 2])$,
$[1\ [1\ [3\ \ [2 \backslash_{1,1,2}\ 2]_{1,3}\ \ 2]\ 2]\ 2]$ has level $\theta(\varepsilon(\varepsilon(\zeta_{\Omega+1}+2)))$,
$[1\ [1\ [1\ \ [2 \backslash_{1,1,2}\ 2]_{1,3}\ \ 3]\ 2]\ 2]$ has level $\theta(\varepsilon(\varepsilon(\zeta_{\Omega+1}2)))$,
$[1\ [1\ [1\ \ [2 \backslash_{1,1,2}\ 2]_{1,3}\ \ 1\ \ [2 \backslash_{1,1,2}\ 2]_{1,3}\ \ 2]\ 2]\ 2]$ has level $\theta(\varepsilon(\varepsilon(\zeta_{\Omega+1}\hat{}2)))$,
$[1\ [1\ [1\ [1\ \ [2 \backslash_{1,1,2}\ 2]_{2,3}\ \ 3]\ 2]\ 2]\ 2]$ has level $\theta(\varepsilon(\varepsilon(\zeta_{\Omega+1}\hat{}\zeta_{\Omega+1})))$,
$[1\ [1\ [1\ [1\ \ [2 \backslash_{1,1,2}\ 2]_{2,3}\ \ 1\ \ [2 \backslash_{1,1,2}\ 2]_{2,3}\ \ 2]\ 2]\ 2]\ 2]$ has level $\theta(\varepsilon(\varepsilon(\zeta_{\Omega+1}\hat{}\zeta_{\Omega+1}\hat{}\zeta_{\Omega+1})))$,
$[1\ [1\ [1\ [1\ [1\ \ [2 \backslash_{1,1,2}\ 2]_{3,3}\ \ 3]\ 2]\ 2]\ 2]\ 2]$ has level $\theta(\varepsilon(\varepsilon(\zeta_{\Omega+1}\hat{}\zeta_{\Omega+1}\hat{}\zeta_{\Omega+1}\hat{}\zeta_{\Omega+1})))$,
$[1\ [1\ [1\ [2\ \ [2 \backslash_{1,1,2}\ 2]_{1,4}\ \ 2]\ 2]\ 2]\ 2]$ has level $\theta(\varepsilon(\varepsilon(\varepsilon(\zeta_{\Omega+1}+1))))$,
$[1\ [1\ [1\ [1\ [2\ \ [2 \backslash_{1,1,2}\ 2]_{1,5}\ \ 2]\ 2]\ 2]\ 2]\ 2]$ has level $\theta(\varepsilon(\varepsilon(\varepsilon(\varepsilon(\zeta_{\Omega+1}+1)))))$,
$[1\ [1\ [1\ [1\ [1\ [2\ \ [2 \backslash_{1,1,2}\ 2]_{1,6}\ \ 2]\ 2]\ 2]\ 2]\ 2]\ 2]$ has level $\theta(\varepsilon(\varepsilon(\varepsilon(\varepsilon(\varepsilon(\zeta_{\Omega+1}+1))))))$.

$[1\ [3 \backslash_{1,1,2}\ 2]\ 2]$ has level $\theta(\zeta_{\Omega+2})$,
$[1\ [4 \backslash_{1,1,2}\ 2]\ 2]$ has level $\theta(\zeta_{\Omega+3})$,
$[1\ [1 \backslash_{1,1,2}\ 3]\ 2]$ has level $\theta(\zeta_{\Omega 2})$,
$[1\ [1 \backslash_{1,1,2}\ 1 \backslash_{1,1,2}\ 2]\ 2]$ has level $\theta(\zeta_{\Omega\hat{}2})$,
$[1\ [1\ [2 \backslash_{2,1,2}\ 2]\ 2]\ 2]$ has level $\theta(\zeta_{\Omega\hat{}\omega})$,
$[1\ [1\ [1 \backslash_{2,1,2}\ 3]\ 2]\ 2]$ has level $\theta(\zeta_{\Omega\hat{}\Omega})$,
$[1\ [1\ [1 \backslash_{2,1,2}\ 1 \backslash_{2,1,2}\ 2]\ 2]\ 2]$ has level $\theta(\zeta_{\Omega\hat{}\Omega\hat{}\Omega})$,
$[1\ [1\ [1\ [1 \backslash_{3,1,2}\ 3]\ 2]\ 2]\ 2]$ has level $\theta(\zeta_{\Omega\hat{}\Omega\hat{}\Omega\hat{}\Omega})$,
$[1\ [1\ [2 \backslash_{1,2,2}\ 2]\ 2]\ 2]$ has level $\theta(\zeta(\varepsilon_{\Omega+1}))$ $(\backslash_{1,n,2} = [1 \backslash_{1,n+1,2}\ 2])$,
$[1\ [1\ [3 \backslash_{1,2,2}\ 2]\ 2]\ 2]$ has level $\theta(\zeta(\varepsilon_{\Omega+2}))$,
$[1\ [1\ [1 \backslash_{1,2,2}\ 3]\ 2]\ 2]$ has level $\theta(\zeta(\varepsilon_{\Omega 2}))$,
$[1\ [1\ [1 \backslash_{1,2,2}\ 1 \backslash_{1,2,2}\ 2]\ 2]\ 2]$ has level $\theta(\zeta(\varepsilon_{\Omega\hat{}2}))$,
$[1\ [1\ [1\ [1 \backslash_{2,2,2}\ 3]\ 2]\ 2]\ 2]$ has level $\theta(\zeta(\varepsilon_{\Omega\hat{}\Omega}))$,
$[1\ [1\ [1\ [1 \backslash_{2,2,2}\ 1 \backslash_{2,2,2}\ 2]\ 2]\ 2]\ 2]$ has level $\theta(\zeta(\varepsilon_{\Omega\hat{}\Omega\hat{}\Omega}))$,
$[1\ [1\ [1\ [1\ [1 \backslash_{3,2,2}\ 3]\ 2]\ 2]\ 2]\ 2]$ has level $\theta(\zeta(\varepsilon_{\Omega\hat{}\Omega\hat{}\Omega\hat{}\Omega}))$,
$[1\ [1\ [1\ [2 \backslash_{1,3,2}\ 2]\ 2]\ 2]\ 2]$ has level $\theta(\zeta(\varepsilon(\varepsilon_{\Omega+1})))$,
$[1\ [1\ [1\ [1\ [2 \backslash_{1,4,2}\ 2]\ 2]\ 2]\ 2]\ 2]$ has level $\theta(\zeta(\varepsilon(\varepsilon(\varepsilon_{\Omega+1}))))$,
$[1\ [1\ [1\ [1\ [1\ [2 \backslash_{1,5,2}\ 2]\ 2]\ 2]\ 2]\ 2]\ 2]$ has level $\theta(\zeta(\varepsilon(\varepsilon(\varepsilon(\varepsilon_{\Omega+1})))))$.

$[1\ [1\ [2 \backslash_{1,1,3}\ 2]\ 2]\ 2]$ has level $\theta(\zeta(\zeta_{\Omega+1}))$ $(\backslash_{1,1,n} = [1 \backslash_{1,1,n+1}\ 2])$,
$[1\ [1\ [3 \backslash_{1,1,3}\ 2]\ 2]\ 2]$ has level $\theta(\zeta(\zeta_{\Omega+2}))$,
$[1\ [1\ [1 \backslash_{1,1,3}\ 3]\ 2]\ 2]$ has level $\theta(\zeta(\zeta_{\Omega 2}))$,
$[1\ [1\ [1 \backslash_{1,1,3}\ 1 \backslash_{1,1,3}\ 2]\ 2]\ 2]$ has level $\theta(\zeta(\zeta_{\Omega\hat{}2}))$,
$[1\ [1\ [1\ [1 \backslash_{2,1,3}\ 3]\ 2]\ 2]\ 2]$ has level $\theta(\zeta(\zeta_{\Omega\hat{}\Omega}))$,
$[1\ [1\ [1\ [1 \backslash_{2,1,3}\ 1 \backslash_{2,1,3}\ 2]\ 2]\ 2]\ 2]$ has level $\theta(\zeta(\zeta_{\Omega\hat{}\Omega\hat{}\Omega}))$,
$[1\ [1\ [1\ [1\ [1 \backslash_{3,1,3}\ 3]\ 2]\ 2]\ 2]\ 2]$ has level $\theta(\zeta(\zeta_{\Omega\hat{}\Omega\hat{}\Omega\hat{}\Omega}))$,
$[1\ [1\ [1\ [2 \backslash_{1,2,3}\ 2]\ 2]\ 2]\ 2]$ has level $\theta(\zeta(\zeta(\varepsilon_{\Omega+1})))$,
$[1\ [1\ [1\ [3 \backslash_{1,2,3}\ 2]\ 2]\ 2]\ 2]$ has level $\theta(\zeta(\zeta(\varepsilon_{\Omega+2})))$,
$[1\ [1\ [1\ [1 \backslash_{1,2,3}\ 3]\ 2]\ 2]\ 2]$ has level $\theta(\zeta(\zeta(\varepsilon_{\Omega 2})))$,

[1 [1 [1 [1 $\backslash_{1,2,3}$ 1 $\backslash_{1,2,3}$ 2] 2] 2] 2] has level θ(ζ(ζ($\varepsilon_{\Omega\wedge 2}$))),
[1 [1 [1 [1 [1 $\backslash_{2,2,3}$ 3] 2] 2] 2] 2] has level θ(ζ(ζ($\varepsilon_{\Omega\wedge\Omega}$))),
[1 [1 [1 [1 [1 $\backslash_{2,2,3}$ 1 $\backslash_{2,2,3}$ 2] 2] 2] 2] 2] has level θ(ζ(ζ($\varepsilon_{\Omega\wedge\Omega\wedge\Omega}$))),
[1 [1 [1 [1 [1 [1 $\backslash_{3,2,3}$ 3] 2] 2] 2] 2] 2] has level θ(ζ(ζ($\varepsilon_{\Omega\wedge\Omega\wedge\Omega\wedge\Omega}$))),
[1 [1 [1 [1 [2 $\backslash_{1,3,3}$ 2] 2] 2] 2] 2] has level θ(ζ(ζ(ε($\varepsilon_{\Omega+1}$)))),
[1 [1 [1 [1 [1 [2 $\backslash_{1,4,3}$ 2] 2] 2] 2] 2] 2] has level θ(ζ(ζ(ε(ε($\varepsilon_{\Omega+1}$))))),
[1 [1 [1 [1 [1 [1 [2 $\backslash_{1,5,3}$ 2] 2] 2] 2] 2] 2] 2] has level θ(ζ(ζ(ε(ε(ε($\varepsilon_{\Omega+1}$)))))),
[1 [1 [1 [2 $\backslash_{1,1,4}$ 2] 2] 2] 2] has level θ(ζ(ζ($\zeta_{\Omega+1}$))),
[1 [1 [1 [1 [2 $\backslash_{1,1,5}$ 2] 2] 2] 2] 2] has level θ(ζ(ζ(ζ($\zeta_{\Omega+1}$)))),
[1 [1 [1 [1 [1 [2 $\backslash_{1,1,6}$ 2] 2] 2] 2] 2] 2] has level θ(ζ(ζ(ζ(ζ($\zeta_{\Omega+1}$))))).

The sequence of separators starting with the last three has limit ordinal θ(φ(3, Ω+1)).

[1 [2 $\backslash_{1,1,1,2}$ 2] 2] has level θ(φ(3, Ω+1)).

Replacing [2 $\backslash_{1,1,2}$ 2] by [2 $\backslash_{1,1,1,2}$ 2] in the list of separators on pages 15-16 entails changing each $\zeta_{\Omega+1}$ in the associated ordinal levels to φ(3, Ω+1). For example,

[1 [2 $\backslash_{1,1,1,2}$ 2] 3] has level θ(φ(3, Ω+1), 1),
[1 [2 $\backslash_{1,1,1,2}$ 2] 1 \ 2] has level θ(φ(3, Ω+1)+1),
[1 [2 $\backslash_{1,1,1,2}$ 2] 1 [2 $\backslash_{1,1,1,2}$ 2] 2] has level θ(φ(3, Ω+1)2),
[1 [1 [2 $\backslash_{1,1,1,2}$ $2]_2$ 3] 2] has level θ(φ(3, Ω+1)^2),
[1 [1 [2 $\backslash_{1,1,1,2}$ $2]_2$ 1 [2 $\backslash_{1,1,1,2}$ $2]_2$ 2] 2] has level θ(φ(3, Ω+1)^φ(3, Ω+1)),
[1 [1 [1 [2 $\backslash_{1,1,1,2}$ $2]_3$ 3] 2] 2] has level θ(φ(3, Ω+1)^φ(3, Ω+1)^φ(3, Ω+1)),
[1 [2 [2 $\backslash_{1,1,1,2}$ $2]_{1,2}$ 2] 2] has level θ(ε(φ(3, Ω+1)+1))   ([2 $\backslash_{1,1,1,2}$ 2] = [1 [2 $\backslash_{1,1,1,2}$ $2]_{1,2}$ 2]),
[1 [1 [2 [2 $\backslash_{1,1,1,2}$ $2]_{1,3}$ 2] 2] 2] has level θ(ε(ε(φ(3, Ω+1)+1))),
[1 [1 [1 [2 [2 $\backslash_{1,1,1,2}$ $2]_{1,4}$ 2] 2] 2] 2] has level θ(ε(ε(ε(φ(3, Ω+1)+1)))).

By substituting [2 $\backslash_{1,1,1,2}$ 2] for the backslash in the list of separators on pages 15-17, each of the associated levels would have the Ω in the ζ function replaced by φ(3, Ω+1). I find that

[1 [2 [2 $\backslash_{1,1,1,2}$ $2]_{1,1,2}$ 2] 2] has level θ(ζ(φ(3, Ω+1)+1))   ([2 $\backslash_{1,1,1,2}$ 2] = [1 [2 $\backslash_{1,1,1,2}$ $2]_{1,1,2}$ 2]),
[1 [2 [2 [2 $\backslash_{1,1,1,2}$ $2]_{1,1,2}$ $2]_{1,2}$ 2] 2] has level θ(ε(ζ(φ(3, Ω+1)+1)+1)),
[1 [1 [2 [2 [2 $\backslash_{1,1,1,2}$ $2]_{1,1,2}$ $2]_{1,3}$ 2] 2] 2] has level θ(ε(ε(ζ(φ(3, Ω+1)+1)+1))),
[1 [3 [2 $\backslash_{1,1,1,2}$ $2]_{1,1,2}$ 2] 2] has level θ(ζ(φ(3, Ω+1)+2)),
[1 [1 [2 $\backslash_{1,1,1,2}$ $2]_{1,1,2}$ 3] 2] has level θ(ζ(φ(3, Ω+1)2)),
[1 [1 [2 $\backslash_{1,1,1,2}$ $2]_{1,1,2}$ 1 [2 $\backslash_{1,1,1,2}$ $2]_{1,1,2}$ 2] 2] has level θ(ζ(φ(3, Ω+1)^2)),
[1 [1 [1 [2 $\backslash_{1,1,1,2}$ $2]_{2,1,2}$ 3] 2] 2] has level θ(ζ(φ(3, Ω+1)^φ(3, Ω+1))),
[1 [1 [2 [2 $\backslash_{1,1,1,2}$ $2]_{1,2,2}$ 2] 2] 2] has level θ(ζ(ε(φ(3, Ω+1)+1))),
[1 [1 [1 [2 [2 $\backslash_{1,1,1,2}$ $2]_{1,3,2}$ 2] 2] 2] 2] has level θ(ζ(ε(ε(φ(3, Ω+1)+1)))),
[1 [1 [2 [2 $\backslash_{1,1,1,2}$ $2]_{1,1,3}$ 2] 2] 2] has level θ(ζ(ζ(φ(3, Ω+1)+1))),
[1 [1 [1 [2 [2 $\backslash_{1,1,1,2}$ $2]_{1,1,4}$ 2] 2] 2] 2] has level θ(ζ(ζ(ζ(φ(3, Ω+1)+1)))).

It follows that

[1 [3 $\backslash_{1,1,1,2}$ 2] 2] has level θ(φ(3, Ω+2)),
[1 [1 $\backslash_{1,1,1,2}$ 3] 2] has level θ(φ(3, Ω2)),
[1 [1 $\backslash_{1,1,1,2}$ 1 $\backslash_{1,1,1,2}$ 2] 2] has level θ(φ(3, Ω^2)),
[1 [1 [1 $\backslash_{2,1,1,2}$ 3] 2] 2] has level θ(φ(3, Ω^Ω)),
[1 [1 [2 $\backslash_{1,2,1,2}$ 2] 2] 2] has level θ(φ(3, $\varepsilon_{\Omega+1}$))   ($\backslash_{1,n,1,2}$ = [1 $\backslash_{1,n+1,1,2}$ 2]),
[1 [1 [1 [2 $\backslash_{1,3,1,2}$ 2] 2] 2] 2] has level θ(φ(3, ε($\varepsilon_{\Omega+1}$))),
[1 [1 [2 $\backslash_{1,1,2,2}$ 2] 2] 2] has level θ(φ(3, $\zeta_{\Omega+1}$))   ($\backslash_{1,1,n,2}$ = [1 $\backslash_{1,1,n+1,2}$ 2]),
[1 [1 [1 [2 $\backslash_{1,1,3,2}$ 2] 2] 2] 2] has level θ(φ(3, ζ($\zeta_{\Omega+1}$))),

$[1\ [1\ [2\ \backslash_{1,1,1,3}\ 2]\ 2]\ 2]$ has level $\theta(\varphi(3, \varphi(3, \Omega+1)))$ $(\backslash_{1,1,1,n} = [1\ \backslash_{1,1,1,n+1}\ 2])$,
$[1\ [1\ [1\ [2\ \backslash_{1,1,1,4}\ 2]\ 2]\ 2]\ 2]$ has level $\theta(\varphi(3, \varphi(3, \varphi(3, \Omega+1))))$,
$[1\ [1\ [1\ [1\ [2\ \backslash_{1,1,1,5}\ 2]\ 2]\ 2]\ 2]\ 2]$ has level $\theta(\varphi(3, \varphi(3, \varphi(3, \varphi(3, \Omega+1)))))$.

The sequence of separators starting with the last three has limit ordinal $\theta(\varphi(4, \Omega+1))$.

$[1\ [2\ \backslash_{1,1,1,1,2}\ 2]\ 2]$ has level $\theta(\varphi(4, \Omega+1))$,
$[1\ [2\ \backslash_{1,1,1,1,1,2}\ 2]\ 2]$ has level $\theta(\varphi(5, \Omega+1))$,
$[1\ [2\ \backslash_{1,1,\dots,1,2}\ 2]\ 2]$ (with n 1's) has level $\theta(\varphi(n, \Omega+1))$.

The limit ordinal of this backslash subscript notation is $\theta(\varphi(\omega, \Omega+1))$.

In general, with k+1 subscripts and $n_1+n_2+\dots+n_k+1-k$ pairs of square brackets ($n_k \geq 2$),

$[S] = [1\ [1\ [1\ [\ \dots\ [1\ [2\ \backslash_{1,n_1,n_2,\dots,n_k}\ 2]\ 2]\ \dots\ ]\ 2]\ 2]\ 2]$
has level $\theta(\varphi(k, \varphi(k, \dots , \varphi(k, \varphi(k\text{-}1, \varphi(k\text{-}1, \dots , \varphi(k\text{-}1, \dots\dots \zeta(\zeta(\dots\zeta(\ \varepsilon(\varepsilon(\dots\varepsilon(\varepsilon_{\Omega+1})\dots\dots))$
(with n_k-1 φ(k's, n_{k-1}-1 φ(k-1's, ... , n_2-1 ζ's and n_1-1 ε's).

Suppose that X is a character string such that [X] is either a single entry array or contains only normal separators in its 'base layer', and has level α – for example, if X = 'm' then α = m-1, if X = '1 [1 \ m] 2' then $\alpha = \varepsilon_{m-2}$. If the first 1 of [S] is replaced by X, we would add α to its ordinal level. If the nth 1 of [S] ($2 \leq n \leq n_1+n_2+\dots+n_k-k$) is replaced by X, its ordinal level would have α added inside the nth outermost layer of brackets, i.e. the ordinal level would be as above but end in '$_{\Omega+1})\dots))+\alpha)\dots))$', with n)'s after '+α'. If the 2 to the left of the backslash in the top layer of [S] is replaced by X ≠ '1', the Ω+1 in the innermost layer of brackets of the ordinal would become Ω+α.

On the other hand, if the rightmost 2 of [S] is replaced by X ≠ '1', the θ function of the ordinal level would have a second argument of α-1 (α < ω) or α (α ≥ ω) instead of 0 (representing the αth 'fixed point'). If the nth rightmost 2 of [S] ($2 \leq n \leq n_1+n_2+\dots+n_k-k$) is replaced by X ≠ '1', the ordinal inside the nth outermost layer of brackets of the ordinal level would be multiplied by α, i.e. the ordinal level would be as above but end in '$_{\Omega+1})\dots))\alpha)\dots))$', with n)'s after 'α'. If the 2 to the right of the backslash and subscripts in the top layer of [S] is replaced by X ≠ '1', the Ω+1 in the innermost layer of the ordinal would become Ωα.

If the $[2\ \backslash_{1,n_1,n_2,\dots,n_k}\ 2]$ at the top layer of [S] is replaced by

$[1\ [1\ [1\ [\ \dots\ [1\ [1\ \backslash_{m,n_1,n_2,\dots,n_k}\ 3]\ 2]\ \dots\ ]\ 2]\ 2]\ 2]$ (m pairs of square brackets, m ≥ 2),

the Ω+1 in the innermost layer of brackets of the associated ordinal level would become Ω^^(2m-2) or a power tower of 2m-2 Ω's. If the $[2\ \backslash_{1,n_1,n_2,\dots,n_k}\ 2]$ is replaced by

$[1\ [1\ [\ \dots\ [1\ [1\ \backslash_{m,n_1,n_2,\dots,n_k}\ 1\ \backslash_{m,n_1,n_2,\dots,n_k}\ 2]\ 2]\ \dots\ ]\ 2]\ 2]$ (m pairs of square brackets, m ≥ 2),

the Ω+1 would become Ω^^(2m-1) or a power tower of 2m-1 Ω's. The limit ordinal substitutes $\varepsilon_{\Omega+1}$ for Ω+1, which is achieved by adding 1 to n_1 or replacing $[2\ \backslash_{1,n_1,n_2,\dots,n_k}\ 2]$ with $[1\ [2\ \backslash_{1,n_1+1,n_2,\dots,n_k}\ 2]\ 2]$.

The recursive definition of an $(n_1, n_2, n_3, \dots , n_k)$-hyperseparator (for k ≥ 1, $n_1 \geq 1$, $n_i \geq 1$ ($1 \leq i < k$) and $n_k \geq 2$ (k ≥ 2)) is that one of the following four conditions hold:

1. It is the $\backslash_{n_1,n_2,n_3,\dots,n_k}$ symbol.
2. Contains at least one $(n_1+1, n_2, n_3, \dots , n_k)$-hyperseparator in its 'base layer'.
3. Subscripted by $n_1,n_2,n_3,\dots,n_i$ and contains at least one $(1, 1, \dots , 1, n_{i+1}+1, n_{i+2}, n_{i+3}, \dots , n_k)$-hyperseparator (with i 1's) in its 'base layer', for some $1 \leq i < k$. (When i = k-1, the hyperseparator level expression would read $(1, 1, \dots , 1, n_k+1)$-hyperseparator (with k-1 1's).)
4. Subscripted by $n_1,n_2,n_3,\dots,n_k$ and contains at least one (1, 1, 1, ... , 1, 2)-hyperseparator (with at least k 1's) in its 'base layer'.

In order to find the minimum number of square brackets around an $(n_1, n_2, n_3, \ldots, n_k)$-hyperseparator necessary to turn it into a normal separator, take the sum of the n_i's (when reduced by 1), then add 1. This is $n_1+n_2+n_3+\ldots+n_k+1-k$ pairs of square brackets. In the case of a backslash with subscript array, take the sum of the subscripts (when reduced by 1), then add 1.

Suppose that M and N are arrays. An M-hyperseparator cannot be on the same 'nested level' as an N-hyperseparator unless both hyperseparators are of the same level (M = N) or one (or both) separators are either normal or 1-hyperseparators (either M = '0', M = '1', N = '0' or N = '1').

Rules A5b and A5b* are modified as follows:-

The backslash in Rule A5b is now the generalised $\backslash_{m_1,m_2,\ldots,m_q}$ symbol, where $q \geq 1$ and $m_q \geq 2$. The $\backslash_j$ in the $R_{n,j-1}$ equation is replaced by $\backslash_{m_1,m_2,\ldots,m_q}$. (Here, the number of subscripts is q, as i, j, k, n and p are already used.)

Rule A5b* (separator $[A_{i,j}(p_{i,j})] = [d\ \#_S]_{m_1,m_2,\ldots,m_k}$, where $d \geq 2$, $k \geq 1$ and $\#_S$ contains at least one $(1, 1, \ldots, 1, r_1, r_2, \ldots)$-hyperseparator (with k 1's) in its base layer, where $r_1 \geq 2$):

$S_{i,j}$ = 'b $\langle A_{i,j}(1)' \rangle$ b $[A_{i,j}(1)]$ b $\langle A_{i,j}(2)' \rangle$ b $[A_{i,j}(2)]$ ... b $\langle A_{i,j}(p_{i,j}-1)' \rangle$ b $[A_{i,j}(p_{i,j}-1)]$
R_b $[d\ \#_S]_{m_1,m_2,\ldots,m_k}$ $c_{i,j}-1\ \#_{i,j}$',

R_n = 'b $\langle R_{n-1} \rangle$ b',

R_1 = 'b $[d-1\ \#_S]_{1,1,\ldots,1,m_k+b-1}$ b' (with k-1 1's in subscript).

In the above subrule, any of the m_i for $1 \leq i \leq k$ and r_i for $i \geq 2$ may take the value 1. Subscripts and hyperseparator levels are written with trailing 1's removed. For example, if $n_k \geq 2$ but $n_i = 1$ for all $i > k$, then $\backslash_{n_1,n_2,\ldots} = \backslash_{n_1,n_2,\ldots,n_k}$, $[X]_{n_1,n_2,\ldots} = [X]_{n_1,n_2,\ldots,n_k}$ and $\langle X \rangle_{n_1,n_2,\ldots} = \langle X \rangle_{n_1,n_2,\ldots,n_k}$ for a string X, and an $(n_1, n_2, \ldots)$-hyperseparator would be an $(n_1, n_2, \ldots, n_k)$-hyperseparator. If $n_i = 1$ for all i, then $\backslash_{n_1,n_2,\ldots} = \backslash$, $[X]_{n_1,n_2,\ldots} = [X]$ and $\langle X \rangle_{n_1,n_2,\ldots} = \langle X \rangle$ for a string X, and an $(n_1, n_2, \ldots)$-hyperseparator would be a 1-hyperseparator.

Note that Rule A5b* with the lowest $(1, 1, \ldots, 1, r_1, r_2, \ldots)$-hyperseparator (with k 1's) within $\#_S$,

$[A_{i,j}(p_{i,j})] = [2\ \backslash_{1,1,\ldots,1,r_1,r_2,\ldots}\ 2]_{m_1,m_2,\ldots,m_k}$ (with k 1's inside brackets)

would mean that

R_1 = 'b $[1\ \backslash_{1,1,\ldots,1,r_1,r_2,\ldots}\ 2]_{1,1,\ldots,1,m_k+b-1}$ b' (k 1's inside and k-1 1's outside brackets)
= 'b $\backslash_{1,1,\ldots,1,m_k+b-1,r_1-1,r_2,\ldots}$ b' (with k-1 1's after backslash).

By setting $m_k = 1$, we obtain

R_1 = 'b $\backslash_{1,1,\ldots,1,b,r_1-1,r_2,\ldots}$ b',

which is how the (k+1)th subscript of the backslash symbol is reduced by 1. (The kth subscript becomes b; all other subscripts remain unchanged.)

In Rule A5b, the road from $R_{n,1}$ to $R_{n-1,1}$ is split up into many parts, via layers of separators. This begins with the i-hyperseparators (i = 2, 3, 4, ...), (i, 2)-hyperseparators, (i, 3)-hyperseparators etc., then the (i, 1, 2)-hyperseparators, (i, 2, 2)-hyperseparators etc., then levels (i, 1, 3), (i, 2, 3) etc., (i, 1, 4), (i, 2, 4) etc., etc., then the 4-entry array levels starting with the (1, 1, 1, 2)-hyperseparators, then the 5-entry array levels, and so on, right up to the final layer of $(m_1, m_2, \ldots, m_q)$-hyperseparators and 1-hyperseparators, with the latter set sandwiched in between the penultimate $(m_1, m_2, \ldots, m_q)$-hyperseparator and the $\backslash_{m_1,m_2,\ldots,m_q}$ symbol. In each case where the array level of the set of hyperseparators begins with 1, this is through a separator carrying a potential subscript – for example, the set of $(1, 1, \ldots, 1, r_1, r_2, \ldots)$-hyperseparators (with k 1's, for some $k \geq 1$, $r_1 \geq 2$, $r_i \geq 1$ for each $i \geq 2$)

is immediately via a separator with an $n_1,n_2,...,n_k$ subscript (for some $n_i \geq 1$ for each $1 \leq i \leq k$, unless every $n_i = 1$).

Separators with subscript arrays give rise to angle brackets with them too. When N = ‘$n_1, n_2, ... , n_p$’ for some $p \geq 1$,

‘$3 \langle k~\#\rangle_N~3$’ = ‘$3 \langle k\text{-}1~\#\rangle_N~3~[k~\#]_N~3 \langle k\text{-}1~\#\rangle_N~3~[k~\#]_N~3 \langle k\text{-}1~\#\rangle_N~3$’ ($k \geq 1$),

‘$a \langle 0~[m~\#^*]~k~\#\rangle_N~b$’ = ‘$a \langle b \langle m\text{-}1~\#^*\rangle~b~[m~\#^*]~k\text{-}1~\#\rangle_N~b$’ ($k \geq 2$),

where #* does not begin with a 2- or higher hyperseparator when m = 1. For the purposes of Rules A2 and A5 (initial part), a ‘2- or higher order hyperseparator’ includes all ($n_1, n_2, ... , n_k$)-hyperseparators, where $n_k \geq 2$. The N can be treated as the rightmost part of the main separator and angle bracket arrays (separated by a ‘superhyperseparator’) in the Angle Bracket Rules.

Subscript arrays can themselves be extended into multidimensional arrays, nested arrays or even as advanced as ordinary separator arrays can go. The single backslash without any subscripts is now equivalent to

$[1 \backslash_{1~[2]~2}~2] = [1 \backslash_{1~[2]~1,2}~2] = [1 \backslash_{1~[2]~1,1,2}~2] = ...$,
$[1 \backslash_{1~[2]~1~[2]~2}~2] = [1 \backslash_{1~[2]~1~[2]~1,2}~2] = [1 \backslash_{1~[2]~1~[2]~1,1,2}~2] = ...$,
$[1 \backslash_{1~[2]~1~[2]~1~[2]~2}~2] = [1 \backslash_{1~[2]~1~[2]~1~[2]~1~[2]~2}~2] = [1 \backslash_{1~[2]~1~[2]~1~[2]~1~[2]~1~[2]~2}~2] = ...$,
$[1 \backslash_{1~[3]~2}~2] = [1 \backslash_{1~[3]~1,2}~2] = ... = [1 \backslash_{1~[3]~1~[2]~2}~2] = [1 \backslash_{1~[3]~1~[2]~1,2}~2] = ...$,
$[1 \backslash_{1~[3]~1~[3]~2}~2] = [1 \backslash_{1~[3]~1~[3]~1~[3]~2}~2] = [1 \backslash_{1~[3]~1~[3]~1~[3]~1~[3]~2}~2] = ...$,
$[1 \backslash_{1~[4]~2}~2] = [1 \backslash_{1~[5]~2}~2] = [1 \backslash_{1~[6]~2}~2] = ...$,
$[1 \backslash_{1~[1,2]~2}~2] = [1 \backslash_{1~[1~[2]~2]~2}~2] = [1 \backslash_{1~[1~[1,2]~2]~2}~2] = ...$,
$[1 \backslash_{1~[1~\backslash~2]~2}~2] = [1 \backslash_{1~[1~\backslash~3]~2}~2] = [1 \backslash_{1~[1~\backslash~4]~2}~2] = ...$,
$[1 \backslash_{1~[1~\backslash~1~\backslash~2]~2}~2] = [1 \backslash_{1~[1~[1~\backslash_2~3]~2]~2}~2] = [1 \backslash_{1~[1~[1~\backslash_2~1~\backslash_2~2]~2]~2}~2] = [1 \backslash_{1~[1~[1~[1~\backslash_3~3]~2]~2]~2}~2] = ...$,
$[1 \backslash_{1~[1~[2~\backslash_{1,2}~2]~2]~2}~2] = [1 \backslash_{1~[1~[2~\backslash_{1,1,2}~2]~2]~2}~2] = [1 \backslash_{1~[1~[2~\backslash_{1,1,1,2}~2]~2]~2}~2] = ...$,
$[1 \backslash_{S_1}~2] = [1 \backslash_{S_2}~2] = [1 \backslash_{S_3}~2] = ...$
(with S_1 = ‘$1~[1~[2~\backslash_{1,2}~2]~2]~2$’ and S_{n+1} = ‘$1~[1~[2~\backslash_{S_n}~2]~2]~2$’).

In fact,

$\backslash = [1 \backslash_{1~[A_1]~1~[A_2]~...~1~[A_k]~2}~2]$,

in all cases where each $[A_i]$ is a normal separator. A subscript array, like a main array (in curly brackets), can only contain normal separators in its ‘base layer’.

An ($n_1~[A_1]~n_2~[A_2]~...~n_k~[A_k]~n_{k+1}$)-hyperseparator (the lowest of which is the $\backslash_{n_1~[A_1]~n_2~[A_2]~...~n_k~[A_k]~n_{k+1}}$ symbol), where each $[A_i]$ is a normal separator, requires a minimum of $n_1+n_2+...+n_{k+1}-k$ pairs of square brackets in order to turn it into a normal separator. This holds true when every $[A_i] = [1]$ (comma).

The generalised backslash subscript array (for normal separators $[A_i]$)

$\backslash_{n_1~[A_1]~n_2~[A_2]~...~n_k~[A_k]~n_{k+1}}$
$= [1 \backslash_{n_1+1~[A_1]~n_2~[A_2]~n_3~[A_3]~n_4~[A_4]~...~n_k~[A_k]~n_{k+1}}~2]$
$= [1 \backslash_{1~[A_1]~n_2+1~[A_2]~n_3~[A_3]~n_4~[A_4]~...~n_k~[A_k]~n_{k+1}}~2]_{n_1}$
$= [1 \backslash_{1~[A_1]~1~[A_2]~n_3+1~[A_3]~n_4~[A_4]~...~n_k~[A_k]~n_{k+1}}~2]_{n_1~[A_1]~n_2}$
$= ......$
$= [1 \backslash_{1~[A_1]~1~[A_2]~...~1~[A_k]~n_{k+1}+1}~2]_{n_1~[A_1]~n_2~[A_2]~...~n_{k-1}~[A_{k-1}]~n_k}$
$= [1 \backslash_{S~[A_1]~n_2~[A_2]~n_3~[A_3]~n_4~[A_4]~...~n_k~[A_k]~n_{k+1}}~2]_{n_1}$ ($[B_1] < [A_1]$)
$= [1 \backslash_{1~[A_1]~S~[A_2]~n_3~[A_3]~n_4~[A_4]...~n_k~[A_k]~n_{k+1}}~2]_{n_1~[A_1]~n_2}$ ($[B_1] < [A_2]$)
$= [1 \backslash_{1~[A_1]~1~[A_2]~S~[A_3]~n_4~[A_4]...~n_k~[A_k]~n_{k+1}}~2]_{n_1~[A_1]~n_2~[A_2]~n_3}$ ($[B_1] < [A_3]$)
$= ......$
$= [1 \backslash_{1~[A_1]~1~[A_2]~...~1~[A_{k-1}]~S~[A_k]~n_{k+1}}~2]_{n_1~[A_1]~n_2~[A_2]~...~n_{k-1}~[A_{k-1}]~n_k}$ ($[B_1] < [A_k]$)
$= [1 \backslash_{1~[A_1]~1~[A_2]~...~1~[A_k]~S}~2]_{n_1~[A_1]~n_2~[A_2]~...~n_k~[A_k]~n_{k+1}}$,

where S = '1 [B_1] 1 [B_2] ... 1 [B_m] 2' (m ≥ 1, [B_i] are normal separators).
The backslash itself in the above equalities (other than the last one) may be substituted by a separator array that contains at least one (1 [A_1] 1 [A_2] ... 1 [A_k] 1 #)-hyperseparator (with # non-empty, containing at least one entry of 2 or greater) in its 'base layer', for example,

$[X \backslash_{1\ [A_1]\ 1\ [A_2]\ \ldots\ 1\ [A_k]\ 1,m}\ Y]n_1\ [A_1]\ n_2\ [A_2]\ \ldots\ n_k\ [A_k]\ n_{k+1}$

$= [1\ \ [X \backslash_{1\ [A_1]\ 1\ [A_2]\ \ldots\ 1\ [A_k]\ 1,m}\ Y]1\ [A_1]\ 1\ [A_2]\ n_3{+}1\ [A_3]\ n_4\ [A_4]\ \ldots\ n_k\ [A_k]\ n_{k+1}\ \ 2]n_1\ [A_1]\ n_2$

(with m ≥ 2; X and Y are strings either side of $\backslash_{1\ [A_1]\ 1\ [A_2]\ \ldots\ 1\ [A_k]\ 1,m}$).

The θ(φ(ω, Ω+1)) level separator

{a, b [1 [2 $\backslash_{1\ [2]\ 2}$ 2] 2] 2} = {a ‹0 [2 $\backslash_{1\ [2]\ 2}$ 2] 2› b}

= {a ‹b ‹b ‹ ... ‹b ‹b $\backslash_{1,1,\ldots,1,b}$ b› b› ... › b› b› b}

(with b pairs of angle brackets and b-1 1's in $\backslash_{1,1,\ldots,1,b}$).

This is equivalent to

{a, b [1 [2 $\backslash_{1,1,1,\ldots,1,2}$ 2] 2] 2} (with b 1's in $\backslash_{1,1,1,\ldots,1,2}$).

With k 1's in the subscript,

{a, b [1 [1 [2 $\backslash_{1,1,1,\ldots,1,2\ [2]\ 2}$ 2] 2] 2] 2} = {a ‹0 [1 [2 $\backslash_{1,1,1,\ldots,1,2\ [2]\ 2}$ 2] 2] 2› b}

= {a ‹b ‹b ‹ ... ‹b ‹b $\backslash_{1,1,\ldots,1,b\ [2]\ 2}$ b› b› ... › b› b› b}

(with b+1 pairs of angle brackets and k-1 1's in $_{1,1,\ldots,1,b}$),

{a, b [1 [2 $\backslash_{1\ [2]\ 1,1,1,\ldots,1,2}$ 2] 2] 2} = {a ‹0 [2 $\backslash_{1\ [2]\ 1,1,1,\ldots,1,2}$ 2] 2› b}

= {a ‹b ‹b ‹ ... ‹b ‹b $\backslash_{1\ [2]\ 1,1,\ldots,1,b}$ b› b› ... › b› b› b}

(with b pairs of angle brackets and k-1 1's in $_{1,1,\ldots,1,b}$).

If there was a chain of 1's and [2]'s prior to the $_{1,1,1,\ldots,1,2}$ in the subscripts of the above two arrays, this chain would remain unchanged prior to the $_{1,1,\ldots,1,b}$ in the subscripts on the right-hand side.

When X is a subscript array beginning with 1 and containing at least one entry of 2 or greater, and n is one more than the sum of all the entries in the 'base layer' of X (when reduced by 1),

{a, b [1 [1 [1 ... [1 [c $\backslash_X$ 2] 2] ... 2] 2] 2] 2} (with n pairs of square brackets)

= {a ‹0 [1 [1 ... [1 [c $\backslash_X$ 2] 2] ... 2] 2] 2› b}

= {a ‹b ‹b ‹ ... ‹b ‹b [c-1 $\backslash_X$ $2]_Y$ b› b› ... › b› b› b} (with n+b-2 pairs of angle brackets),

for some other subscript array Y (where [X] > [Y]). If c = 2, the separator [1 $\backslash_X$ $2]_Y$ 'drops down' to $\backslash_Z$ (where Z is another subscript array) since the first non-1 entry in X is beyond the entry corresponding to the final non-1 entry in Y. Z would inherit all the entries and separators of Y plus the entries and separators of X from the first separator in X that has no counterpart in Y, onwards (with the first non-1 entry of X reduced by 1). This implies that Y and Z would be identical when X contains a single non-1 entry of 2.

I find that

[1 [2 $\backslash_{1\ [2]\ 2}$ 2] 2] has level θ(φ(ω, Ω+1)),

[1 [2 $\backslash_{1\ [2]\ 2}$ 2] 3] has level θ(φ(ω, Ω+1), 1),

[1 [2 $\backslash_{1\ [2]\ 2}$ 2] 1 \ 2] has level θ(φ(ω, Ω+1)+1),

[1 [2 $\backslash_{1\ [2]\ 2}$ 2] 1 [2 $\backslash_{1\ [2]\ 2}$ 2] 2] has level θ(φ(ω, Ω+1)2),

[1 [1 [2 $\backslash_{1\ [2]\ 2}$ $2]_2$ 3] 2] has level θ(φ(ω, Ω+1)^2),

[1 [1 [2 $\backslash_{1\ [2]\ 2}$ $2]_2$ 1 [2 $\backslash_{1\ [2]\ 2}$ $2]_2$ 2] 2] has level θ(φ(ω, Ω+1)^φ(ω, Ω+1)),

[1 [1 [1 [2 $\backslash_{1\ [2]\ 2}$ $2]_3$ 3] 2] 2] has level θ(φ(ω, Ω+1)^φ(ω, Ω+1)^φ(ω, Ω+1)),

[1 [2 [2 $\backslash_{1\ [2]\ 2}$ $2]_{1,2}$ 2] 2] has level θ(ε(φ(ω, Ω+1)+1)) ([2 $\backslash_{1\ [2]\ 2}$ 2] = [1 [2 $\backslash_{1\ [2]\ 2}$ $2]_{1,2}$ 2]),

[1 [1 [2 [2 $\backslash_{1\ [2]\ 2}$ $2]_{1,3}$ 2] 2] 2] has level θ(ε(ε(φ(ω, Ω+1)+1))),

[1 [1 [1 [2 [2 $\backslash_{1\ [2]\ 2}$ $2]_{1,4}$ 2] 2] 2] 2] has level θ(ε(ε(ε(φ(ω, Ω+1)+1)))).

Since $[2 \backslash_{1\,[2]\,2}\, 2] = [1\ [2 \backslash_{1\,[2]\,2}\, 2]_{1,1,\ldots,1,2}\ 2]$, for any finite number of 1's in $_{1,1,\ldots,1,2}$,

$[1\ [2\ \ [2 \backslash_{1\,[2]\,2}\, 2]_{1,1,2}\ \ 2]\ 2]$ has level $\theta(\zeta(\varphi(\omega, \Omega+1)+1))$,

$[1\ [2\ \ [2 \backslash_{1\,[2]\,2}\, 2]_{1,1,\ldots,1,2}\ \ 2]\ 2]$ (with n 1's in $_{1,1,\ldots,1,2}$) has level $\theta(\varphi(n, \varphi(\omega, \Omega+1)+1))$,

and as $\varphi(n, \varphi(\omega, \alpha)+1)$ has limit ordinal $\varphi(\omega, \alpha+1)$ as $n \to \omega$, it follows that

$[1\ [3 \backslash_{1\,[2]\,2}\, 2]\ 2]$ has level $\theta(\varphi(\omega, \Omega+2))$,

$[1\ [4 \backslash_{1\,[2]\,2}\, 2]\ 2]$ has level $\theta(\varphi(\omega, \Omega+3))$,

$[1\ [1 \backslash_{1\,[2]\,2}\, 3]\ 2]$ has level $\theta(\varphi(\omega, \Omega 2))$,

$[1\ [1 \backslash_{1\,[2]\,2}\, 1 \backslash_{1\,[2]\,2}\, 2]\ 2]$ has level $\theta(\varphi(\omega, \Omega\text{^}2))$,

$[1\ [1\ [1 \backslash_{2\,[2]\,2}\, 3]\ 2]\ 2]$ has level $\theta(\varphi(\omega, \Omega\text{^}\Omega))$ $(\backslash_{n\,[2]\,2} = [1 \backslash_{n+1\,[2]\,2}\, 2])$,

$[1\ [1\ [2 \backslash_{1,2\,[2]\,2}\, 2]\ 2]\ 2]$ has level $\theta(\varphi(\omega, \varepsilon_{\Omega+1}))$ $(\backslash_{1,n\,[2]\,2} = [1 \backslash_{1,n+1\,[2]\,2}\, 2])$,

$[1\ [1\ [1\ [2 \backslash_{1,3\,[2]\,2}\, 2]\ 2]\ 2]\ 2]$ has level $\theta(\varphi(\omega, \varepsilon(\varepsilon_{\Omega+1})))$,

$[1\ [1\ [2 \backslash_{1,1,2\,[2]\,2}\, 2]\ 2]\ 2]$ has level $\theta(\varphi(\omega, \zeta_{\Omega+1}))$ $(\backslash_{1,1,n\,[2]\,2} = [1 \backslash_{1,1,n+1\,[2]\,2}\, 2])$,

$[1\ [1\ [2 \backslash_{1,1,\ldots,1,2\,[2]\,2}\, 2]\ 2]\ 2]$ (with n 1's in $_{1,1,\ldots,1,2}$) has level $\theta(\varphi(\omega, \varphi(n, \Omega+1)))$,

$[1\ [1\ [2 \backslash_{1\,[2]\,3}\, 2]\ 2]\ 2]$ has level $\theta(\varphi(\omega, \varphi(\omega, \Omega+1)))$ $(\backslash_{1\,[2]\,n} = [1 \backslash_{1\,[2]\,n+1}\, 2])$,

$[1\ [1\ [1\ [2 \backslash_{1\,[2]\,4}\, 2]\ 2]\ 2]\ 2]$ has level $\theta(\varphi(\omega, \varphi(\omega, \varphi(\omega, \Omega+1))))$,

$[1\ [2 \backslash_{1\,[2]\,1,2}\, 2]\ 2]$ has level $\theta(\varphi(\omega+1, \Omega+1))$,

$[1\ [2 \backslash_{1\,[2]\,1,1,\ldots,1,2}\, 2]\ 2]$ (with n 1's in $_{1,1,\ldots,1,2}$) has level $\theta(\varphi(\omega+n, \Omega+1))$,

$[1\ [2 \backslash_{1\,[2]\,1\,[2]\,2}\, 2]\ 2]$ has level $\theta(\varphi(\omega 2, \Omega+1))$,

$[1\ [2 \backslash_{1\,[2]\,1\,[2]\,1\,[2]\,2}\, 2]\ 2]$ has level $\theta(\varphi(\omega 3, \Omega+1))$,

$[1\ [2 \backslash_{1\,[2]\,1\,[2]\,\ldots\,1\,[2]\,2}\, 2]\ 2]$ (with n [2]'s) has level $\theta(\varphi(\omega n, \Omega+1))$.

The most significant higher separators are as follows:-

$[1\ [2 \backslash_{1\,[3]\,2}\, 2]\ 2]$ has level $\theta(\varphi(\omega\text{^}2, \Omega+1))$,

$[1\ [2 \backslash_{1\,[4]\,2}\, 2]\ 2]$ has level $\theta(\varphi(\omega\text{^}3, \Omega+1))$,

$[1\ [2 \backslash_{1\,[1,2]\,2}\, 2]\ 2]$ has level $\theta(\varphi(\omega\text{^}\omega, \Omega+1))$,

$[1\ [2 \backslash_{1\,[1\,[2]\,2]\,2}\, 2]\ 2]$ has level $\theta(\varphi(\omega\text{^}\omega\text{^}\omega, \Omega+1))$,

$[1\ [2 \backslash_{1\,[1\,[1,2]\,2]\,2}\, 2]\ 2]$ has level $\theta(\varphi(\omega\text{^}\omega\text{^}\omega\text{^}\omega, \Omega+1))$,

$[1\ [2 \backslash_{1\,[1\,\backslash\,2]\,2}\, 2]\ 2]$ has level $\theta(\varphi(\varepsilon_0, \Omega+1))$,

$[1\ [2 \backslash_{1\,[1\,\backslash\,3]\,2}\, 2]\ 2]$ has level $\theta(\varphi(\varepsilon_1, \Omega+1))$,

$[1\ [2 \backslash_{1\,[1\,\backslash\,1\,\backslash\,2]\,2}\, 2]\ 2]$ has level $\theta(\varphi(\zeta_0, \Omega+1))$,

$[1\ [2 \backslash_{1\,[1\,\backslash\,1\,\backslash\,1\,\backslash\,2]\,2}\, 2]\ 2]$ has level $\theta(\varphi(\varphi(3, 0), \Omega+1))$,

$[1\ [2 \backslash_{1\,[1\,[2\,\backslash_2\,2]\,2]\,2}\, 2]\ 2]$ has level $\theta(\varphi(\varphi(\omega, 0), \Omega+1))$,

$[1\ [2 \backslash_{1\,[1\,[1\,\backslash_2\,3]\,2]\,2}\, 2]\ 2]$ has level $\theta(\varphi(\Gamma_0, \Omega+1))$,

$[1\ [2 \backslash_{1\,[1\,[1\,\backslash_2\,1\,\backslash_2\,2]\,2]\,2}\, 2]\ 2]$ has level $\theta(\varphi(\theta(\Omega\text{^}\Omega), \Omega+1))$,

$[1\ [2 \backslash_{1\,[1\,[1\,[1\,\backslash_3\,3]\,2]\,2]\,2}\, 2]\ 2]$ has level $\theta(\varphi(\theta(\Omega\text{^}\Omega\text{^}\Omega), \Omega+1))$,

$[1\ [2 \backslash_{1\,[1\,[2\,\backslash_{1,2}\,2]\,2]\,2}\, 2]\ 2]$ has level $\theta(\varphi(\theta(\varepsilon_{\Omega+1}), \Omega+1))$,

$[1\ [2 \backslash_{1\,[1\,[2\,\backslash_{1,1,2}\,2]\,2]\,2}\, 2]\ 2]$ has level $\theta(\varphi(\theta(\zeta_{\Omega+1}), \Omega+1))$,

$[1\ [2 \backslash_{1\,[1\,[2\,\backslash_{1\,[2]\,2}\,2]\,2]\,2}\, 2]\ 2]$ has level $\theta(\varphi(\theta(\varphi(\omega, \Omega+1)), \Omega+1))$,

$[1\ [2 \backslash_{1\,[1\,[2\,\backslash_{1\,[1\,\backslash\,2]\,2}\,2]\,2]\,2}\, 2]\ 2]$ has level $\theta(\varphi(\theta(\varphi(\varepsilon_0, \Omega+1)), \Omega+1))$.

Taking S_1 = '$1\ [1\ [2 \backslash_{1,2}\, 2]\ 2]\ 2$' and S_{n+1} = '$1\ [1\ [2 \backslash_{S_n}\, 2]\ 2]\ 2$',

$[1\ [2 \backslash_{S_1}\, 2]\ 2]$ has level $\theta(\varphi(\theta(\varepsilon_{\Omega+1}), \Omega+1))$,

$[1\ [2 \backslash_{S_2}\, 2]\ 2]$ has level $\theta(\varphi(\theta(\varphi(\theta(\varepsilon_{\Omega+1}), \Omega+1)), \Omega+1))$,

$[1\ [2 \backslash_{S_3}\, 2]\ 2]$ has level $\theta(\varphi(\theta(\varphi(\theta(\varphi(\theta(\varepsilon_{\Omega+1}), \Omega+1)), \Omega+1)), \Omega+1))$.

The limit ordinal of the S_n sequence, and the nested backslash subscript array notation – which I will call the Nested Subscript Array Notation – is $\theta(\varphi(\Omega, 1))$. This is due to the fact that $\varphi(\Omega, 1)$ is the limit of $\varphi(\alpha, \Omega+1) = \varphi(\alpha, \varphi(\Omega, 0)+1)$ as $\alpha \to \Omega$.

A second (more powerful) ordinal collapsing function (θ_1) can be used within the θ function in order to represent the ordinals above the Bachmann-Howard ordinal. It works as follows:

$$\begin{aligned}
&\theta_1(0, \alpha) = \Omega^\wedge\alpha, \\
&\theta_1(1, \alpha) = \varepsilon_{\Omega+1+\alpha}, \\
&\theta_1(2, \alpha) = \zeta_{\Omega+1+\alpha}, \\
&\theta_1(\alpha) = \theta_1(\alpha, 0), \\
&\theta_1(\alpha, \beta) = \varphi(\alpha, \Omega+1+\beta) && (1 \le \alpha < \Omega) \\
&\qquad\quad = \varphi(\alpha, 1+\beta) && (\alpha = \Omega) \\
&\qquad\quad = \varphi(\alpha, \beta) && (\Omega < \alpha < \Omega_2), \\
&\theta_1(\alpha+1) = \theta_1(\alpha, \theta_1(\alpha, \theta_1(\alpha, \ldots \theta_1(\alpha)\ldots))) && (\text{with } \omega\ \theta_1\text{'s}), \\
&\theta_1(\Omega) = \varphi(\Omega, 1) = \theta_1(\theta(\theta_1(\theta(\theta_1(\ldots\theta(\theta_1(0))\ldots))))) && (\text{with } \omega\ \theta_1\text{'s}), \\
&\theta_1(\Omega_2) = \Gamma_{\Omega+1} = \theta_1(\theta_1(\theta_1(\ldots\theta_1(0)\ldots))) && (\text{with } \omega\ \theta_1\text{'s}), \\
&\theta(\theta_1(\alpha, \beta)) = \theta(\alpha, \beta) && (\alpha \ge \Omega_2),
\end{aligned}$$

where Ω_2 denotes the second uncountable ordinal. ($\Omega = \Omega_1$ is the first uncountable ordinal.) We can define higher collapsing functions (θ_n) and create higher uncountable ordinals (Ω_{n+1}) by analogy with θ_1 and Ω_2 above, for example,

$$\begin{aligned}
&\theta_n(0, \alpha) = \Omega_n^\wedge\alpha, \\
&\theta_n(\alpha) = \theta_n(\alpha, 0), \\
&\theta_n(\alpha, \beta) = \varphi(\alpha, \Omega_n+1+\beta) && (1 \le \alpha < \Omega_n) \\
&\qquad\quad = \varphi(\alpha, 1+\beta) && (\alpha = \Omega_n) \\
&\qquad\quad = \varphi(\alpha, \beta) && (\Omega_n < \alpha < \Omega_{n+1}), \\
&\theta_n(\alpha+1) = \theta_n(\alpha, \theta_n(\alpha, \theta_n(\alpha, \ldots \theta_1(\alpha)\ldots))) && (\text{with } \omega\ \theta_n\text{'s}), \\
&\theta_n(\Omega) = \theta_n(\theta(\theta_n(\theta(\theta_n(\ldots\theta(\theta_n(0))\ldots))))) && (\text{with } \omega\ \theta_n\text{'s}), \\
&\theta_n(\Omega_{k+1}) = \theta_n(\theta_k(\theta_n(\theta_k(\theta_n(\ldots\theta_k(\theta_n(0))\ldots))))) && (\text{with } \omega\ \theta_n\text{'s}, k \le n), \\
&\theta_n(\Omega_{n+1}) = \theta_n(\theta_n(\theta_n(\ldots\theta_n(0)\ldots))) && (\text{with } \omega\ \theta_n\text{'s}), \\
&\theta(\theta_n(\alpha, \beta)) = \theta(\alpha, \beta) && (\alpha \ge \Omega_{n+1}), \\
&\theta_k(\theta_n(\alpha, \beta)) = \theta_k(\alpha, \beta) && (\alpha \ge \Omega_{n+1}, k < n).
\end{aligned}$$

We can extend this to θ_α functions and Ω_α uncountable ordinals for transfinite α. $\theta(\Omega_\omega)$ is a special ordinal since it is the proof theoretic ordinal of the subsystem $\Pi^1_1\text{-CA}_0$ of second-order arithmetic. There is a whole new universe of ordinals that are far greater than the first fixed point of $\alpha = \Omega_\alpha$ within the θ function.

An alternative (single-argument) ordinal collapsing function (ψ) works as follows:

$$\begin{aligned}
&\psi(\alpha) = \varepsilon_\alpha && (\alpha < \Omega), \\
&\psi(\alpha+1) = \psi(\alpha)^{\wedge\wedge}\omega && (\text{power tower of } \psi(\alpha)\text{'s of height } \omega), \\
&\psi((\Omega^\wedge\alpha)\beta) = \varphi(1+\alpha, \beta-1) && (1 \le \alpha < \Omega, 1 \le \beta < \omega) \\
&\qquad\qquad = \varphi(1+\alpha, \beta) && (1 \le \alpha < \Omega, \omega \le \beta < \Omega) \\
&\qquad\qquad = \theta(1+\alpha, \beta-1) && (\alpha \ge 1, 1 \le \beta < \omega) \\
&\qquad\qquad = \theta(1+\alpha, \beta) && (\alpha \ge 1, \omega \le \beta < \Omega).
\end{aligned}$$

Like the θ_1 function (within θ), we can define the ψ_1 function (within ψ) in order to proceed beyond the Bachmann-Howard ordinal. It works as follows:

$$\begin{aligned}
&\psi_1(\alpha) = \varepsilon_{\Omega+1+\alpha} && (\alpha < \Omega_2), \\
&\psi_1(\alpha+1) = \psi_1(\alpha)^{\wedge\wedge}\omega && (\text{power tower of } \psi_1(\alpha)\text{'s of height } \omega), \\
&\psi_1((\Omega_2^\wedge\alpha)\beta) = \varphi(1+\alpha, \Omega+\beta) && (1 \le \alpha < \Omega, 1 \le \beta < \Omega_2) \\
&\qquad\qquad = \varphi(\alpha, \beta-1) && (\Omega < \alpha < \Omega_2, 1 \le \beta < \omega) \\
&\qquad\qquad = \varphi(\alpha, \beta) && (\Omega < \alpha < \Omega_2, \omega \le \beta < \Omega_2 \text{ or } \alpha = \Omega, 1 \le \beta < \Omega_2) \\
&\qquad\qquad = \theta_1(1+\alpha, \beta-1) && (\alpha \ge 1, 1 \le \beta < \omega) \\
&\qquad\qquad = \theta_1(1+\alpha, \beta) && (\alpha \ge 1, \omega \le \beta < \Omega_2).
\end{aligned}$$

This can be extended further by defining higher ψ_α functions which are comparable with the θ_α functions. I prefer to use the θ system of collapsing functions since these begin with the finite numbers (rather than ε_0) and it is more closely related to the Veblen function. Also, a power tower of Ω's within the θ function contains one fewer Ω than the equivalent power tower of Ω's within the ψ function.

The ordinal $\theta(\varphi(\Omega, 1)) = \theta(\theta_1(\Omega)) = \psi_1(\Omega_2\text{^}\Omega)$, using the θ_1 and ψ_1 functions as defined above.

The θ(φ(ω^2, Ω+1)) level separator

$\{a, b\ [1\ [2\ \backslash_{1\ [3]\ 2}\ 2]\ 2]\ 2\} = \{a\ \langle 0\ [2\ \backslash_{1\ [3]\ 2}\ 2]\ 2\rangle\ b\}$

$= \{a\ \langle b\ \langle b\ \langle\ \ldots\ \langle b\ \langle b\ \backslash_{1\ \langle 2\rangle\ b\ (\leftarrow b)}\ b\rangle\ b\rangle\ \ldots\ \rangle\ b\rangle\ b\rangle\ b\}$

(with b pairs of angle brackets outside subscript),

where the (←b) means replace the final entry by b. Since

‘1 ‹2› b (←b)’ = ‘1 ‹1› b [2] 1 ‹1› b [2] ... 1 ‹1› b [2] 1 ‹1› b (←b)’ (with b-1 [2]'s)

= ‘1,1,..,1 [2] 1,1,..,1 [2] .... 1,1,..,1 [2] 1,1,..,1,b’ (b-1 1's after final [2])

= ‘1 [2] 1 [2] ... 1 [2] 1,1,...,1,b’ (remove trailing 1's),

it follows that

$\{a, b\ [1\ [2\ \backslash_{1\ [3]\ 2}\ 2]\ 2]\ 2\} = \{a\ \langle b\ \langle b\ \langle\ \ldots\ \langle b\ \langle b\ \backslash_{1\ [2]\ 1\ [2]\ \ldots\ 1\ [2]\ 1,1,\ldots,1,b}\ b\rangle\ b\rangle\ \ldots\ \rangle\ b\rangle\ b\rangle\ b\}$

(b pairs of angle brackets, b-1 ‘1 [2]'s and b-1 1's in $_{1,1,\ldots,1,b}$).

The following additions are made to Rule A1:

‘a ‹0› b (←c)’ = ‘c’,

‘a ‹1› b (←c)’ = ‘a, a, ... , a, c’ (with b-1 a's).

The following additions are made to Rule A2:

‘a ‹0 #› b (←c)’ = ‘c’,

‘a ‹1 #› b (←c)’ = ‘a [1 #] a [1 #] ... a [1 #] c’ (with b-1 a's),

where # begins with a 2- or higher order hyperseparator.

The θ(φ(ω^ω, Ω+1)) level separator

$\{a, b\ [1\ [2\ \backslash_{1\ [1,2]\ 2}\ 2]\ 2]\ 2\} = \{a\ \langle 0\ [2\ \backslash_{1\ [1,2]\ 2}\ 2]\ 2\rangle\ b\}$

$= \{a\ \langle b\ \langle b\ \langle\ \ldots\ \langle b\ \langle b\ \backslash_{1\ \langle 0,2\rangle\ b\ (\leftarrow b)}\ b\rangle\ b\rangle\ \ldots\ \rangle\ b\rangle\ b\rangle\ b\}$

$= \{a\ \langle b\ \langle b\ \langle\ \ldots\ \langle b\ \langle b\ \backslash_{1\ \langle b\rangle\ b\ (\leftarrow b)}\ b\rangle\ b\rangle\ \ldots\ \rangle\ b\rangle\ b\rangle\ b\}$

(with b pairs of angle brackets outside subscript),

where ‘1 ‹b› b (←b)’ = ‘1 ‹b-1› b [b] 1 ‹b-1› b [b] ... 1 ‹b-1› b [b] 1 ‹b-1› b (←b)’ (with b-1 [b]'s)

= ‘1 [b] 1 [b] ... 1 [b] 1 [b-1] 1 [b-1] ... 1 [b-1] ...... 1 [2] 1 [2] ... 1 [2] 1,1,...,1,b’

(b-1 each of [b]'s, [b-1]'s, ... , [2]'s and b-1 1's after final [2]).

Examples of arrays with separators of higher levels are as follows:-

$\{a, b\ [1\ [2\ \backslash_{1\ [1\ \backslash\ 2]\ 2}\ 2]\ 2]\ 2\}$

$= \{a\ \langle 0\ [2\ \backslash_{1\ [1\ \backslash\ 2]\ 2}\ 2]\ 2\rangle\ b\}$

$= \{a\ \langle b\ \langle b\ \langle\ \ldots\ \langle b\ \langle b\ \backslash_{1\ \langle 0\ \backslash\ 2\rangle\ b\ (\leftarrow b)}\ b\rangle\ b\rangle\ \ldots\ \rangle\ b\rangle\ b\rangle\ b\}$

(with b pairs of angle brackets outside subscript)

$= \{a\ \langle b\ \langle b\ \langle\ \ldots\ \langle b\ \langle b\ \backslash_{1\ \langle b\ \langle b\ \langle\ \ldots\ \langle b\ \langle b\rangle\ b\rangle\ \ldots\ \rangle\ b\rangle\ b\rangle\ b\ (\leftarrow b)}\ b\rangle\ b\rangle\ \ldots\ \rangle\ b\rangle\ b\rangle\ b\}$

(with b-1 pairs of angle brackets in subscript),

$\{a, b\ [1\ [2\ \backslash_{1\ [1\ [1\ \backslash_2\ 3]\ 2]\ 2}\ 2]\ 2]\ 2\}$

$= \{a\ \langle 0\ [2\ \backslash_{1\ [1\ [1\ \backslash_2\ 3]\ 2]\ 2}\ 2]\ 2\rangle\ b\}$

$= \{a\ \langle b\ \langle b\ \langle\ \ldots\ \langle b\ \langle b\ \backslash_{1\ \langle 0\ [1\ \backslash_2\ 3]\ 2\rangle\ b\ (\leftarrow b)}\ b\rangle\ b\rangle\ \ldots\ \rangle\ b\rangle\ b\rangle\ b\}$

(with b pairs of angle brackets outside subscript)

$= \{a \langle b \langle b \langle \ldots \langle b \langle b \backslash_{1\ \langle b \langle b \langle b \langle \ldots \langle b \langle b \backslash_2 2\rangle\ b \backslash_2 2\rangle \ldots \rangle\ b \backslash_2 2\rangle\ b \backslash_2 2\rangle\ b\rangle\ b\ (\leftarrow b)} b\rangle\ b\rangle \ldots \rangle\ b\rangle\ b\rangle\ b\}$

(with b pairs of angle brackets in subscript),

$\{a, b\ [1\ [2 \backslash_{1\ [1\ [2 \backslash_{1,2} 2]\ 2]\ 2}\ 2]\ 2]\ 2\}$

$= \{a \langle 0\ [2 \backslash_{1\ [1\ [2 \backslash_{1,2} 2]\ 2]\ 2}\ 2]\ 2\rangle\ b\}$

$= \{a \langle b \langle b \langle \ldots \langle b \langle b \backslash_{1\ \langle 0\ [2 \backslash_{1,2} 2]\ 2\rangle\ b\ (\leftarrow b)} b\rangle\ b\rangle \ldots \rangle\ b\rangle\ b\rangle\ b\}$

(with b pairs of angle brackets outside subscript)

$= \{a \langle b \langle b \langle \ldots \langle b \langle b \backslash_{1\ \langle b \langle b \langle \ldots \langle b \langle b \backslash_b b\rangle\ b\rangle \ldots \rangle\ b\rangle\ b\rangle\ b\ (\leftarrow b)} b\rangle\ b\rangle \ldots \rangle\ b\rangle\ b\rangle\ b\}$

(with b pairs of angle brackets in subscript).

In general, when each $[A_i]$ is a normal separator,

$\{a, b\ [1\ [2 \backslash_{1\ [A_1]\ 1\ [A_2] \ldots 1\ [A_k]\ 2}\ 2]\ 2]\ 2\}$

$= \{a \langle 0\ [2 \backslash_{1\ [A_1]\ 1\ [A_2] \ldots 1\ [A_k]\ 2}\ 2]\ 2\rangle\ b\}$

$= \{a \langle b \langle b \langle \ldots \langle b \langle b \backslash_{1\ [A_1]\ 1\ [A_2] \ldots 1\ [A_{k-1}]\ 1 \langle A_k' \rangle\ b\ (\leftarrow b)} b\rangle\ b\rangle \ldots \rangle\ b\rangle\ b\rangle\ b\}$

(with b pairs of angle brackets and k 1's),

where A_k' is identical to A_k except that the first entry is reduced by 1.

Taking the arbitrary string S = '1 $[B_1]$ 1 $[B_2]$ ... 1 $[B_m]$ 2', where $m \geq 1$ and each $[B_i]$ is a normal separator, the recursive definition of an (n_1 $[A_1]$ n_2 $[A_2]$ ... n_k $[A_k]$ n_{k+1})-hyperseparator (for $k \geq 0$, $n_1 \geq 1$, $n_i \geq 1$ ($1 \leq i \leq k$), $n_{k+1} \geq 2$ ($k \geq 1$) and normal separators $[A_i]$) is that one of the following four conditions hold:

1. It is the $\backslash_{n_1 [A_1] n_2 [A_2] \ldots n_k [A_k] n_{k+1}}$ symbol.
2. Contains at least one (n_1+1 $[A_1]$ n_2 $[A_2]$ n_3 $[A_3]$ ... n_k $[A_k]$ n_{k+1})-hyperseparator in its 'base layer'.
3. Subscripted by n_1 $[A_1]$ n_2 $[A_2]$ ... n_{i-1} $[A_{i-1}]$ n_i and contains at least one (1 $[A_1]$ 1 $[A_2]$ ... 1 $[A_i]$ n_{i+1}+1 $[A_{i+1}]$ n_{i+2} $[A_{i+2}]$ ... n_k $[A_k]$ n_{k+1})-hyperseparator or (1 $[A_1]$ 1 $[A_2]$ ... 1 $[A_{i-1}]$ S $[A_i]$ n_{i+1} $[A_{i+1}]$ n_{i+2} $[A_{i+2}]$ ... n_k $[A_k]$ n_{k+1})-hyperseparator in its 'base layer', with $[B_1] < [A_i]$, for some $1 \leq i \leq k$. (When i = k, the hyperseparator expressions would read (1 $[A_1]$ 1 $[A_2]$ ... 1 $[A_k]$ n_{k+1}+1)-hyperseparator or (1 $[A_1]$ 1 $[A_2]$ ... 1 $[A_{k-1}]$ S $[A_k]$ n_{k+1})-hyperseparator.)
4. Subscripted by n_1 $[A_1]$ n_2 $[A_2]$ ... n_k $[A_k]$ n_{k+1} and contains at least one (1 $[A_1]$ 1 $[A_2]$ ... 1 $[A_k]$ S)-hyperseparator in its 'base layer'.

The highest order hyperseparators in an array are backslash symbols, so these are determined first, followed by the next highest hyperseparators, and so on, down to the 1-hyperseparators. A normal separator is a separator that cannot be defined using the above recursive definition – it is neither subscripted nor a backslash symbol, and it contains no 2- or higher order hyperseparators in its 'base layer'.

In Rule A5, subrules b*, c, d and e now become subrules c, d, e and f respectively. The backslash in Rule A5b is now the generalised subscript array $\backslash_M$ symbol, where M contains at least one entry of 2 or greater. The modified and complete Angle Bracket Rules are as follows:-

Rule A1 (only 1 entry of either 0 or 1):

'$a \langle 0 \rangle_N b$' = 'a',

'$a \langle 1 \rangle_N b$' = '$a, a, \ldots, a$' (with b a's),

'$a \langle 0 \rangle_N b\ (\leftarrow c)$' = '$c$',

'$a \langle 1 \rangle_N b\ (\leftarrow c)$' = '$a, a, \ldots, a, c$' (with b-1 a's).

Rule A2 (only 1 entry of either 0 or 1 prior to 2-hyperseparator or higher order hyperseparator):

‘a ‹0 #›$_N$ b’ = ‘a’,

‘a ‹1 #›$_N$ b’ = ‘a [1 #]$_N$ a [1 #]$_N$ ... [1 #]$_N$ a’ (with b a’s),

‘a ‹0 #›$_N$ b (←c)’ = ‘c’,

‘a ‹1 #›$_N$ b (←c)’ = ‘a [1 #]$_N$ a [1 #]$_N$ ... a [1 #]$_N$ c’ (with b-1 a’s),

where # begins with a 2- or higher order hyperseparator.

Rule A3 (last entry in any 1-space or higher dimensional space of array is 1):

‘a ‹# [A] 1›$_N$ b’ = ‘a ‹#›$_N$ b’.

When [A] is an M_1-hyperseparator, [B] is an M_2-hyperseparator and $M_1 < M_2$, or $M_1 = M_2$ and level of [A] is less than level of [B],

‘a ‹# [A] 1 [B] #*›$_N$ b’ = ‘a ‹# [B] #*›$_N$ b’.

Remove trailing 1’s.

Rule A4 (number to right of angle brackets is 1):

‘a ‹A›$_N$ 1’ = ‘a’.

Rule A5 (Rules A1-4 do not apply, first entry is 0, separator immediately prior to next non-1 entry ($c_{1,1}$) is [$A_{1,1}(p_{1,1})$]):

‘a ‹ 0 [$A_{1,1}(1)$] 1 [$A_{1,1}(2)$] ... 1 [$A_{1,1}(p_{1,1}-1)$] 1 [$A_{1,1}(p_{1,1})$] $c_{1,1}$ #$_{1,1}$ #* ›$_N$ b’ = ‘a ‹$S_{1,1}$ #*›$_N$ b’,

where $p_{1,1} \geq 1$, each of [$A_{1,1}(i^*)$] is either normal separator or 1-hyperseparator, and #* is either an empty string or begins with a 2- or higher order hyperseparator.

Set i = 1 and j = 1, and follow Rules A5a-f (a, b, c and f are terminal, d and e are not).

Rule A5a (separator [$A_{i,1}(p_{i,1})$] = [1 \$_2$ 2] = \):

$S_{i,1}$ = ‘R_b’,

R_n = ‘b ‹$A_{i,1}(1)$’› b [$A_{i,1}(1)$] b ‹$A_{i,1}(2)$’› b [$A_{i,1}(2)$] ... b ‹$A_{i,1}(p_{i,1}-1)$’› b [$A_{i,1}(p_{i,1}-1)$]
b ‹R_{n-1}› b \ $c_{i,1}$-1 #$_{i,1}$’,

R_1 = ‘0’.

Rule A5b (Rule A5a does not apply, separator [$A_{i,j}(p_{i,j})$] = \$_M$, where $j \geq 2$ and M ≠ ‘1’):

$S_{i,j}$ = ‘$R_{b,j-1}$’,

$R_{n,j-1}$ = ‘b ‹$A_{i,j}(1)$’› b [$A_{i,j}(1)$] b ‹$A_{i,j}(2)$’› b [$A_{i,j}(2)$] ... b ‹$A_{i,j}(p_{i,j}-1)$’› b [$A_{i,j}(p_{i,j}-1)$]
b ‹$A_{i,1}(1)$’› b [$A_{i,1}(1)$] b ‹$A_{i,1}(2)$’› b [$A_{i,1}(2)$] ... b ‹$A_{i,1}(p_{i,1}-1)$’› b [$A_{i,1}(p_{i,1}-1)$]
b ‹$R_{n-1,1}$› b [$A_{i,1}(p_{i,1})$] $c_{i,1}$-1 #$_{i,1}$ \$_M$ $c_{i,j}$-1 #$_{i,j}$’,

$R_{n,k}$ = ‘b ‹$A_{i,k+1}(1)$’› b [$A_{i,k+1}(1)$] b ‹$A_{i,k+1}(2)$’› b [$A_{i,k+1}(2)$] ...
b ‹$A_{i,k+1}(p_{i,k+1}-1)$’› b [$A_{i,k+1}(p_{i,k+1}-1)$] b ‹$R_{n,k+1}$› b [$A_{i,k+1}(p_{i,k+1})$] $c_{i,k+1}$-1 #$_{i,k+1}$’ ($1 \leq k < j-1$),

$R_{1,1}$ = ‘0’.

Rule A5c (Rules A5a-b do not apply, separator [$A_{i,j}(p_{i,j})$] = [d #$_1$]$_M$, where $d \geq 2$ and #$_1$ contains at least one H-hyperseparator in its base layer, where H begins with 1 and H ≠ ‘1’; in other words,

H = ‘1 [H_1] 1 [H_2] ... 1 [H_k] h #$_2$’,

where $h \geq 2$, $k \geq 1$ and each of [H_i] is a normal separator):

$S_{i,j}$ = ‘b ‹$A_{i,j}(1)$’› b [$A_{i,j}(1)$] b ‹$A_{i,j}(2)$’› b [$A_{i,j}(2)$] ... b ‹$A_{i,j}(p_{i,j}-1)$’› b [$A_{i,j}(p_{i,j}-1)$] R_b [d #$_1$]$_M$ $c_{i,j}$-1 #$_{i,j}$’,

R_n = ‘b ‹R_{n-1}› b’,

R_1 = ‘b [d-1 #$_1$]$_{1\ [H_1]\ 1\ [H_2]\ \ldots\ 1\ [H_{k-1}]\ 1\ \langle H_k' \rangle\ b\ (\leftarrow m+b-1)}$ b’,

where m (which may be 1) is the kth and final entry in the subscript array M when written as

M = ‘m_1 [H_1] m_2 [H_2] ... m_{k-1} [H_{k-1}] m’.

Rule A5d (Rules A5a-c do not apply, separator

$[A_{i,j}(p_{i,j})] = [1\ [A_{i,j+1}(1)]\ 1\ [A_{i,j+1}(2)]\ ...\ 1\ [A_{i,j+1}(p_{i,j+1})]$
$1\ [A_{i+1,1}(1)]\ 1\ [A_{i+1,1}(2)]\ ...\ 1\ [A_{i+1,1}(p_{i+1,1})]\ c_{i+1,1}\ \#_{i+1,1}]$,

where $p_{i,j+1} \geq 1$, $p_{i+1,1} \geq 1$, $c_{i+1,1} \geq 2$, each of $[A_{i+1,1}(i^*)]$ is either a normal separator or 1-hyperseparator, and each of $[A_{i,j+1}(i^*)]$ is a 2- or higher order hyperseparator):

$S_{i,j}$ = 'b ‹$A_{i,j}(1)$'› b $[A_{i,j}(1)]$ b ‹$A_{i,j}(2)$'› b $[A_{i,j}(2)]$ ... b ‹$A_{i,j}(p_{i,j}-1)$'› b $[A_{i,j}(p_{i,j}-1)]$
b ‹T_i› b $[A_{i,j}(p_{i,j})]$ $c_{i,j}-1$ $\#_{i,j}$',

T_i = 'b ‹$A_{i,j+1}(1)$'› b $[A_{i,j+1}(1)]$ b ‹$A_{i,j+1}(2)$'› b $[A_{i,j+1}(2)]$ ... b ‹$A_{i,j+1}(p_{i,j+1})$'› b $[A_{i,j+1}(p_{i,j+1})]$ $S_{i+1,1}$'.

Increment i by 1, reset j = 1, and repeat Rules A5a-f.

Rule A5e (Rules A5a-d do not apply, separator

$[A_{i,j}(p_{i,j})] = [1\ [A_{i,j+1}(1)]\ 1\ [A_{i,j+1}(2)]\ ...\ 1\ [A_{i,j+1}(p_{i,j+1})]\ c_{i,j+1}\ \#_{i,j+1}]$,

where $p_{i,j+1} \geq 1$, $c_{i,j+1} \geq 2$ and each of $[A_{i,j+1}(i^*)]$ is a 2- or higher order hyperseparator):

$S_{i,j}$ = 'b ‹$A_{i,j}(1)$'› b $[A_{i,j}(1)]$ b ‹$A_{i,j}(2)$'› b $[A_{i,j}(2)]$ ... b ‹$A_{i,j}(p_{i,j}-1)$'› b $[A_{i,j}(p_{i,j}-1)]$
b ‹$S_{i,j+1}$› b $[A_{i,j}(p_{i,j})]$ $c_{i,j}-1$ $\#_{i,j}$'.

Increment j by 1 and repeat Rules A5a-f.

Rule A5f (Rules A5a-e do not apply):

$S_{i,j}$ = 'b ‹$A_{i,j}(1)$'› b $[A_{i,j}(1)]$ b ‹$A_{i,j}(2)$'› b $[A_{i,j}(2)]$ ... b ‹$A_{i,j}(p_{i,j})$'› b $[A_{i,j}(p_{i,j})]$ $c_{i,j}-1$ $\#_{i,j}$'.

Rule A6 (Rules A1-5 do not apply):

'a ‹n #›$_N$ b' = 'a ‹n-1 #›$_N$ b [n #]$_N$ a ‹n-1 #›$_N$ b [n #]$_N$ ... [n #]$_N$ a ‹n-1 #›$_N$ b'
(with b 'a ‹n-1 #›$_N$ b' strings).

Notes:

1. A, B, $A_{i,j}(1)$, $A_{i,j}(2)$, ... , $A_{i,j}(p_{i,j})$, H_i are strings of characters within separators.
2. $A_{i,j}(1)$', $A_{i,j}(2)$', ... , $A_{i,j}(p_{i,j})$', H_k' are strings of characters within angle brackets that are identical to the strings $A_{i,j}(1)$, $A_{i,j}(2)$, ... , $A_{i,j}(p_{i,j})$, H_k respectively except that the first entries of each have been reduced by 1. If $A_{i,j}(i^*)$ (for some $1 \leq i^* \leq p_{i,j}$) begins with 1, $A_{i,j}(i^*)$' begins with 0.
3. M and N are strings of characters that make up subscript arrays.
4. $S_{i,j}$, T_i, R_n and $R_{n,k}$ are string building functions which create strings of characters. The R functions involve nesting the same string of characters around itself n-1 times before being replaced by the string '0'.
5. #, #*, $\#_{i,j}$, $\#_1$ and $\#_2$ are strings of characters representing the remainder of the array (can be null or empty).
6. A $\backslash_N$ symbol is an N-hyperseparator. A recursive definition of hyperseparators is given on page 25. A separator that contains no 2- or higher order hyperseparators in its 'base layer' is a normal separator (or 0-hyperseparator).
7. The comma is used as shorthand for the [1] separator.
8. Any 2- or higher hyperseparator may carry a subscript. For example, in Rule A5b, each of the $[A_{i,j}(i^*)]$ separators, for $1 \leq i^* < p_{i,j}$, is (like $[A_{i,j}(p_{i,j})] = \backslash_M$) an M-hyperseparator (where M is at least '2' as Rule A5a does not apply), and may either carry the subscript M(i*) (either the whole of M or the left part of M up to a certain entry) or no subscript at all. If $[A_{i,j}(i^*)]$, for some $1 \leq i^* < p_{i,j}$, has the subscript M(i*), it is written $[A_{i,j}(i^*)]_{M(i^*)}$, and the associated angle bracket array that replaces the preceding 1 would be 'b ‹$A_{i,j}(i^*)$'›$_{M(i^*)}$ b'.
9. The $[1\ \backslash_M\ 2]_N$ separator (M ≠ '1') reduces to a $\backslash_X$ symbol, for a subscript array X, according to a special rule (see below).

Whenever we encounter a separator of the form $[1 \backslash_X 2]_Y$ (with a backslash symbol between the only two entries of 1 and 2), where X and Y are subscript arrays and X ≠ '1', this separator 'drops down' to a simple backslash symbol of the form $\backslash_Z$, where Z is another subscript array. This special rule is known as the Dropping Down Rule.

Dropping Down Rule (separator is of the form $[1 \backslash_M 2]_N$ with M ≠ '1', i.e. has only two entries of 1 and 2 to left and right respectively of backslash subscript array, which is a 2- or higher order hyperseparator):

$$[1\ \backslash_{1\ [A_1]\ 1\ [A_2]\ \ldots\ 1\ [A_m]\ n_{m+1}\ \#}\ 2]_{n_1\ [A_1]\ n_2\ [A_2]\ \ldots\ n_{k-1}\ [A_{k-1}]\ n_k}$$
$$= \backslash_{n_1\ [A_1]\ n_2\ [A_2]\ \ldots\ n_k\ [A_k]\ 1\ [A_{k+1}]\ 1\ [A_{k+2}]\ \ldots\ 1\ [A_m]\ n_{m+1}-1\ \#},$$

where $0 \le k \le m$, $n_{m+1} \ge 2$, each of $[A_i]$ is a normal separator and # represents the remainder of the backslash subscript array.

When k = m, the Dropping Down Rule becomes:

$$[1\ \backslash_{1\ [A_1]\ 1\ [A_2]\ \ldots\ 1\ [A_m]\ n_{m+1}\ \#}\ 2]_{n_1\ [A_1]\ n_2\ [A_2]\ \ldots\ n_{m-1}\ [A_{m-1}]\ n_m}$$
$$= \backslash_{n_1\ [A_1]\ n_2\ [A_2]\ \ldots\ n_m\ [A_m]\ n_{m+1}-1\ \#}.$$

When k = 0, the Dropping Down Rule becomes:

$$[1\ \backslash_{1\ [A_1]\ 1\ [A_2]\ \ldots\ 1\ [A_m]\ n_{m+1}\ \#}\ 2]$$
$$= \backslash_{1\ [A_1]\ 1\ [A_2]\ \ldots\ 1\ [A_m]\ n_{m+1}-1\ \#}.$$

When k = m = 0, the Dropping Down Rule becomes:

$$[1\ \backslash_{n_1\ \#}\ 2] = \backslash_{n_1-1\ \#}.$$

Trailing 1's in subscript arrays are removed as with separator arrays and angle bracket arrays. For example,

$$\backslash_{\#\ [A]\ 1} = \backslash_{\#} \text{ and } [X]_{\#\ [A]\ 1} = [X]_{\#}.$$

When [A] < [B],

$$\backslash_{\#\ [A]\ 1\ [B]\ \#^*} = \backslash_{\#\ [B]\ \#^*} \text{ and } [X]_{\#\ [A]\ 1\ [B]\ \#^*} = [X]_{\#\ [B]\ \#^*}.$$

This is my complete Nested Subscript Array Notation. The limit ordinal of this notation is $\theta(\varphi(\Omega, 1))$ or $\theta(\theta_1(\Omega))$.

In Rule A5b, the pathway from $R_{n,1}$ to $R_{n-1,1}$ is split up into many parts, via layers of separators. $R_{n,1}$ represents the first layer of 2-hyperseparators or $(1\ [B_1]\ 1\ [B_2]\ \ldots\ 1\ [B_q]\ 2)$-hyperseparators (for normal separators $[B_i]$); $R_{n,2}$ comprises the second layer of separators, which could be the set of 3-hyperseparators, (1, 2)-hyperseparators, (2, 2)-hyperseparators, (1, 3)-hyperseparators, or $(n_1\ [B_1]\ n_2\ [B_2]\ \ldots\ n_q\ [B_q]\ n_{q+1})$-hyperseparators, where $n_1+n_2+\ldots+n_{q+1}-q$ is either 2 or 3 (but not 2-hyperseparators); $R_{n,3}$ contains the third layer of $(n_1\ [B_1]\ n_2\ [B_2]\ \ldots\ n_q\ [B_q]\ n_{q+1})$-hyperseparators, where $n_1+n_2+\ldots+n_{q+1}-q$ is either 2, 3 or 4 (but not 2- or 3-hyperseparators). In general, $R_{n,k}$ is the string of $(n_1\ [B_1]\ n_2\ [B_2]\ \ldots\ n_q\ [B_q]\ n_{q+1})$-hyperseparators, where $n_1+n_2+\ldots+n_{q+1}-q$ can be any integer from 2 to k+1 (but not k-hyperseparators or below). The final layer of M-hyperseparators and 1-hyperseparators is represented by $R_{n,j-1}$, which sandwiches the 1-hyperseparators in between the penultimate M-hyperseparator and the $\backslash_M$ symbol. In each case where the array level of the set of hyperseparators begins with 1, this is through a separator which may possess a subscript – for example, the set of $(1\ [B_1]\ 1\ [B_2]\ \ldots\ 1\ [B_q]\ n_{q+1}\ \#)$-hyperseparators (for some $q \ge 1$ and $n_{q+1} \ge 2$) is immediately via a separator with an $n_1\ [B_1]\ n_2\ [B_2]\ \ldots\ n_{r-1}\ [B_{r-1}]\ n_r$ subscript (for some $0 \le r \le q$, where r = 0 represents no subscript).

Rule A5c with the lowest H-hyperseparator within $\#_1$,

$$[A_{i,j}(p_{i,j})] = [2 \backslash_{1\,[H_1]\,1\,[H_2]\,\ldots\,1\,[H_k]\,h\,\#_2}\ 2]_M$$

would mean that

$$R_1 = \text{‘}b\ \ [1 \backslash_{1\,[H_1]\,1\,[H_2]\,\ldots\,1\,[H_k]\,h\,\#_2}\ 2]_{1\,[H_1]\,1\,[H_2]\,\ldots\,1\,[H_{k-1}]\,1\,\langle H_k'\rangle\,b\,(\leftarrow m+b-1)}\ \ b\text{’}$$

$$= \text{‘}b\ \ \backslash_{1\,[H_1]\,1\,[H_2]\,\ldots\,1\,[H_{k-1}]\,1\,\langle H_k'\rangle\,b\,(\leftarrow m+b-1)\,[H_k]\,h-1\,\#_2}\ \ b\text{’} \qquad \text{(Dropping Down Rule).}$$

By setting m = 1, we obtain

$$R_1 = \text{‘}b\ \ \backslash_{1\,[H_1]\,1\,[H_2]\,\ldots\,1\,[H_{k-1}]\,1\,\langle H_k'\rangle\,b\,(\leftarrow b)\,[H_k]\,h-1\,\#_2}\ \ b\text{’},$$

which is how the (k+1)th entry of the backslash subscript array is reduced by 1. (The kth entry is completely filled up with 1's in the space corresponding to the separator $[H_k]$, with b 1's in each ‘row’, b ‘rows’ in each ‘plane’, etc., and the very last of these 1's is replaced by b; all other entries remain unchanged.)

Can anyone beat this function? It is defined as follows:

$$S(n) = \{3, n\ \ [1\ [2 \backslash_{R_n} 2]\ 2]\ \ 2\},$$

where $R_i = \text{‘}1\ [1\ [2 \backslash_{R_{i-1}} 2]\ 2]\ 2\text{’}$,

$R_1 = \text{‘}1, 2\text{’}$.

While S(1) = 3, the number

$$S(2) = \{3, 2\ \ [1\ [2 \backslash_{1\,[1\,[2\,\backslash_{1,2}\,2]\,2]\,2}\ 2]\ 2]\ \ 2\}$$

$$= \{3\ \langle 0\ [2 \backslash_{1\,[1\,[2\,\backslash_{1,2}\,2]\,2]\,2}\ 2]\ 2\rangle\ 2\}$$

$$= \{3\ \langle 2\ \langle 2 \backslash_A 2\rangle\ 2\rangle\ 2\},$$

where $A = \text{‘}1\ \langle 0\ [2 \backslash_{1,2} 2]\ 2\rangle\ 2\ (\leftarrow 2)\text{’}$

$= \text{‘}1\ \langle 2\ \langle 2 \backslash_2 2\rangle\ 2\rangle\ 2\ (\leftarrow 2)\text{’}$

$= \text{‘}1\ \langle 2 \backslash 2\ [2 \backslash_2 2]\ 2 \backslash 2\rangle\ 2\ (\leftarrow 2)\text{’}$

$= \text{‘}1\ [2\backslash 2\ [2 \backslash_2 2]\ 2\backslash 2]\ 1\ [1\backslash 2\ [2 \backslash_2 2]\ 2\backslash 2]\ 1\ [2\ [2 \backslash_2 2]\ 2\backslash 2]\ 1\ [1\ [2 \backslash_2 2]\ 2\backslash 2]$

$1\ [2\backslash 2\ [2 \backslash_2 2]\ 1\backslash 2]\ 1\ [1\backslash 2\ [2 \backslash_2 2]\ 1\backslash 2]\ 1\ [2\ [2 \backslash_2 2]\ 1\backslash 2]\ 1\ [1\ [2 \backslash_2 2]\ 1\backslash 2]$

$1\ [2\backslash 2\ [2 \backslash_2 2]\ 2]\ 1\ [1\backslash 2\ [2 \backslash_2 2]\ 2]\ 1\ [2\ [2 \backslash_2 2]\ 2]\ 1\ [1\ [2 \backslash_2 2]\ 2]\ 1\ [2\backslash 2]\ 1\ [1\backslash 2]\ 1\ [2]\ 1,2\text{’}$,

which means that

$$S(2) = \{3\ \langle 2\ [1 \backslash_A 2]\ 2\ [2 \backslash_A 2]\ 2\ [1 \backslash_A 2]\ 2\rangle\ 2\}$$

$$= \{3\ \langle 2 \backslash 2\ [2 \backslash_A 2]\ 2 \backslash 2\rangle\ 2\} \qquad ([1 \backslash_A 2] = \backslash \text{ as A has single non-1 entry of 2})$$

$$= \{B\ [1\backslash 2\ [2 \backslash_A 2]\ 2\backslash 2]\ B\ [2\backslash 2\ [2 \backslash_A 2]\ 2\backslash 2]\ B\ [1\backslash 2\ [2 \backslash_A 2]\ 2\backslash 2]\ B\},$$

where $B = \text{‘}3\ \langle 0 \backslash 2\ [2 \backslash_A 2]\ 2 \backslash 2\rangle\ 2\text{’}$

$= \text{‘}3\ \langle 2\ [2 \backslash_A 2]\ 2 \backslash 2\rangle\ 2\text{’}$

$= \text{‘}C\ [1\ [2 \backslash_A 2]\ 2\backslash 2]\ C\ [2\ [2 \backslash_A 2]\ 2\backslash 2]\ C\ [1\ [2 \backslash_A 2]\ 2\backslash 2]\ C\text{’}$,

where $C = \text{‘}3\ \langle 0\ [2 \backslash_A 2]\ 2 \backslash 2\rangle\ 2\text{’}$

$= \text{‘}3\ \langle 2\ \langle 2 \backslash_{A^*} 2\rangle\ 2\ [2 \backslash_A 2]\ 1 \backslash 2\rangle\ 2\text{’}$,

where A* is identical to A except that the final ‘1,2’ is replaced by $\text{‘}1\ \langle 0\rangle\ 2\ (\leftarrow 2)\text{’} = \text{‘}2\text{’}$ (Rule A5c), which means that

$$C = \text{‘}3\ \langle 2\ [1 \backslash_{A^*} 2]\ 2\ [2 \backslash_{A^*} 2]\ 2\ [1 \backslash_{A^*} 2]\ 2\ [2 \backslash_A 2]\ 1 \backslash 2\rangle\ 2\text{’}$$

$$= \text{‘}3\ \langle 2 \backslash 2\ [2 \backslash_{A^*} 2]\ 2 \backslash 2\ [2 \backslash_A 2]\ 1 \backslash 2\rangle\ 2\text{’} \qquad ([1 \backslash_{A^*} 2] = \backslash \text{ as A* has single non-1 entry).}$$

It would be rather tedious to go any further – we would eventually encounter a string where there are 65,536 $[2 \backslash_{A_i} 2]$ separators in

$$\text{‘}3\ \langle 2 \backslash 2\ [2 \backslash_2 2]\ 2 \backslash 2\ [2 \backslash_{1,2} 2]\ 1 \backslash 2\ [2 \backslash_{1\,[2]\,2} 2]\ 1 \backslash 2\ \ldots\ [2 \backslash_{A^*} 2]\ 1 \backslash 2\ [2 \backslash_A 2]\ 1 \backslash 2\rangle\ 2\text{’},$$

where each A_{i+1} is identical to A_i (starting with A_1 = A and A_2 = A*) except that the $\text{‘}1\ [X_{i+1}]\ 2\text{’}$ at the end of A_{i+1} (taking $[X_{i+1}]$ as its final separator) is replaced by $\text{‘}1\ \langle X_{i+1}'\rangle\ 2\ (\leftarrow 2)\text{’}$ (Rule A5c), before the last A_i contains a single entry.

The third value of the S function,

$$S(3) = \{3, 3\ \ [1\ [2 \backslash_{1\,[1\,[2\,\backslash_A\,2]\,2]\,2}\ 2]\ 2]\ \ 2\} \qquad (\text{with } A = \text{‘}1\ [1\ [2 \backslash_{1,2} 2]\ 2]\ 2\text{’})$$

$= \{3 \langle 0\ [2 \backslash_{1\ [1\ [2 \backslash_{A}\ 2]\ 2]\ 2}\ 2]\ 2\rangle\ 3\}$
$= \{3 \langle 3 \langle 3 \langle 3 \backslash_{B}\ 3\rangle\ 3\rangle\ 3\rangle\ 3\}$,

where $B =$ ‘$1 \langle 0\ [2 \backslash_{A}\ 2]\ 2\rangle\ 3\ (\leftarrow 3)$’
$=$ ‘$1 \langle 0\ [2 \backslash_{1\ [1\ [2 \backslash_{1,2}\ 2]\ 2]\ 2}\ 2]\ 2\rangle\ 3\ (\leftarrow 3)$’
$=$ ‘$1 \langle 3 \langle 3 \langle 3 \backslash_{1 \langle 0\ [2 \backslash_{1,2}\ 2]\ 2\rangle\ 3\ (\leftarrow 3)}\ 3\rangle\ 3\rangle\ 3\rangle\ 3\ (\leftarrow 3)$’
$=$ ‘$1 \langle 3 \langle 3 \langle 3 \backslash_{1 \langle 3 \langle 3 \langle 3 \backslash_{3}\ 3\rangle\ 3\rangle\ 3\rangle\ 3\ (\leftarrow 3)}\ 3\rangle\ 3\rangle\ 3\rangle\ 3\ (\leftarrow 3)$’.

In general,

$S(n) = \{3 \langle n \langle n \langle \ldots \langle n \langle n \backslash_{R_n}\ n\rangle\ n\rangle \ldots \rangle\ n\rangle\ n\rangle\ n\}$ (with n pairs of angle brackets),

where $R_i =$ ‘$1 \langle n \langle n \langle \ldots \langle n \langle n \backslash_{R_{i-1}}\ n\rangle\ n\rangle \ldots \rangle\ n\rangle\ n\rangle\ n\ (\leftarrow n)$’ (with n pairs of angle brackets),
$R_1 =$ ‘n’.

Imagine how huge this number must be:

$S(S(S(\ldots S(3) \ldots)))$ (with $S(3)$ S’s).

It surely must be scraping infinity!

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