Beyond Bird's Nested Arrays III

Using the single-argument theta function for expressing ordinals (e.g. θ(1) = ε_0, θ(2) = ζ_0, θ(Ω) = Γ_0), the most significant separators introduced so far are:-

[1] has level 0,
[2] has level 1,
[1, 2] has level ω,
[1 [2] 2] has level ω^ω,
[1 [1, 2] 2] has level ω^ω^ω,
[1 [1 [2] 2] 2] has level ω^ω^ω^ω,
[1 \ 2] has level θ(1) (\ is shorthand for [1 ¬ 2]),
[1 [1 \ 2] 2 \ 2] has level θ(1)2,
[1 [1 [1 \ 2] 2 \ 2] 2 \ 2] has level θ(1)^θ(1),
[1 \ 3] has level θ(1, 1) = θ(1)^^ω (using the two-argument function, θ(α, 0) = θ(α)),
[1 \ 4] has level θ(1, 2) = θ(1, 1)^^ω,
[1 \ 1, 2] has level θ(1, ω),
[1 \ 1 [1 \ 2] 2] has level θ(1, θ(1)),
[1 \ 1 \ 2] has level θ(2) = θ(1, θ(1, θ(1, ... θ(1)...))) (with ω θ's),
[1 \ 1 \ 1 \ 2] has level θ(3) = θ(2, θ(2, θ(2, ... θ(2)...))) (with ω θ's),
[1 [2 ¬ 2] 2] has level θ(ω),
[1 [1, 2 ¬ 2] 2] has level θ(ω^ω),
[1 [1 \ 2 ¬ 2] 2] has level θ(θ(1)),
[1 [1 [1 \ 2 ¬ 2] 2 ¬ 2] 2] has level θ(θ(θ(1))),
[1 [1 ¬ 3] 2] has level θ(Ω),
[1 [2 ¬ 3] 2] has level θ(Ωω),
[1 [1 \ 2 ¬ 3] 2] has level θ(Ωθ(1)),
[1 [1 [1 ¬ 3] 2 ¬ 3] 2] has level θ(Ωθ(Ω)),
[1 [1 ¬ 4] 2] has level θ(Ω^2),
[1 [1 ¬ 1, 2] 2] has level θ(Ω^ω) (small Veblen ordinal),
[1 [1 ¬ 1 [2] 2] 2] has level θ(Ω^ω^ω),
[1 [1 ¬ 1 \ 2] 2] has level θ(Ω^θ(1)),
[1 [1 ¬ 1 [2 ¬ 2] 2] 2] has level θ(Ω^θ(ω)),
[1 [1 ¬ 1 [1 ¬ 3] 2] 2] has level θ(Ω^θ(Ω)),
[1 [1 ¬ 1 [1 ¬ 1, 2] 2] 2] has level θ(Ω^θ(Ω^ω)),
[1 [1 ¬ 1 [1 ¬ 1 \ 2] 2] 2] has level θ(Ω^θ(Ω^θ(1))),
[1 [1 ¬ 1 [1 ¬ 1 [1 ¬ 1 \ 2] 2] 2] 2] has level θ(Ω^θ(Ω^θ(Ω^θ(1)))),
[1 [1 ¬ 1 ¬ 2] 2] has level θ(Ω^Ω) (large Veblen ordinal).

If [X] and [Y] are hyperseparators (\ or arrays separated by ¬ symbols) such that [1 [X] 2] and [1 [Y] 2] have levels of θ(α) and θ(β) respectively (where α ≥ β), then [1 [X] 1 [Y] 2] would have level θ(α+β).

[1 [1 ¬ 1 ¬ 2] 1 \ 2] has level θ(Ω^Ω+1),
[1 [1 ¬ 1 ¬ 2] 1 [1 ¬ 3] 2] has level θ(Ω^Ω+Ω),
[1 [1 ¬ 1 ¬ 2] 1 [1 ¬ 1, 2] 2] has level θ(Ω^Ω+Ω^ω),
[1 [1 ¬ 1 ¬ 2] 1 [1 ¬ 1 \ 2] 2] has level θ(Ω^Ω+Ω^θ(1)),
[1 [1 ¬ 1 ¬ 2] 1 [1 ¬ 1 [1¬3] 2] 2] has level θ(Ω^Ω+Ω^θ(Ω)),
[1 [1 ¬ 1 ¬ 2] 1 [1 ¬ 1 ¬ 2] 2] has level θ((Ω^Ω)2),
[1 [2 ¬ 1 ¬ 2] 2] has level θ((Ω^Ω)ω),
[1 [3 ¬ 1 ¬ 2] 2] has level θ((Ω^Ω)ω^2),

[1 [1, 2 ¬ 1 ¬ 2] 2] has level θ((Ω^Ω)ω^ω),
[1 [1 \ 2 ¬ 1 ¬ 2] 2] has level θ((Ω^Ω)θ(1)),
[1 [1 [1¬3] 2 ¬ 1 ¬ 2] 2] has level θ((Ω^Ω)θ(Ω)),
[1 [1 [1¬1,2] 2 ¬ 1 ¬ 2] 2] has level θ((Ω^Ω)θ(Ω^ω)),
[1 [1 [1 ¬ 1 \ 2] 2 ¬ 1 ¬ 2] 2] has level θ((Ω^Ω)θ(Ω^θ(1))),
[1 [1 [1 ¬ 1 [1¬3] 2] 2 ¬ 1 ¬ 2] 2] has level θ((Ω^Ω)θ(Ω^θ(Ω))),
[1 [1 [1 ¬ 1 ¬ 2] 2 ¬ 1 ¬ 2] 2] has level θ((Ω^Ω)θ(Ω^Ω)),
[1 [1 [1 [1¬1¬2] 2 ¬1¬2] 2 ¬1¬2] 2] has level θ((Ω^Ω)θ((Ω^Ω)θ(Ω^Ω))),
[1 [1 [1 [1 [1¬1¬2] 2 ¬1¬2] 2 ¬1¬2] 2 ¬1¬2] 2] has level θ((Ω^Ω)θ((Ω^Ω)θ((Ω^Ω)θ(Ω^Ω)))).

[1 [1 ¬ 2 ¬ 2] 2] has level θ(Ω^(Ω+1)),
[1 [1 ¬ 3 ¬ 2] 2] has level θ(Ω^(Ω+2)),
[1 [1 ¬ 1, 2 ¬ 2] 2] has level θ(Ω^(Ω+ω)),
[1 [1 ¬ 1 \ 2 ¬ 2] 2] has level θ(Ω^(Ω+θ(1))),
[1 [1 ¬ 1 [1¬3] 2 ¬ 2] 2] has level θ(Ω^(Ω+θ(Ω))),
[1 [1 ¬ 1 [1¬1,2] 2 ¬ 2] 2] has level θ(Ω^(Ω+θ(Ω^ω))),
[1 [1 ¬ 1 [1 ¬ 1 \ 2] 2 ¬ 2] 2] has level θ(Ω^(Ω+θ(Ω^θ(1)))),
[1 [1 ¬ 1 [1 ¬ 1 [1¬3] 2] 2 ¬ 2] 2] has level θ(Ω^(Ω+θ(Ω^θ(Ω)))),
[1 [1 ¬ 1 [1 ¬ 1 ¬ 2] 2 ¬ 2] 2] has level θ(Ω^(Ω+θ(Ω^Ω))),
[1 [1¬ 1 [1¬ 1 [1¬1¬2] 2 ¬2] 2 ¬2] 2] has level θ(Ω^(Ω+θ(Ω^(Ω+θ(Ω^Ω))))),
[1 [1¬ 1 [1¬ 1 [1¬ 1 [1¬1¬2] 2 ¬2] 2 ¬2] 2 ¬2] 2] has level
θ(Ω^(Ω+θ(Ω^(Ω+θ(Ω^(Ω+θ(Ω^Ω))))))).

[1 [1 ¬ 1 ¬ 3] 2] has level θ(Ω^(Ω2)),
[1 [1 ¬ 1 ¬ 4] 2] has level θ(Ω^(Ω3)),
[1 [a+1 ¬ b+1 ¬ c+1] 2] (where c ≥ 1) has level θ((Ω^(Ωc+b))ω^a),
[1 [1 ¬ 1 ¬ 1, 2] 2] has level θ(Ω^(Ωω)),
[1 [1 ¬ 1 ¬ 1 \ 2] 2] has level θ(Ω^(Ωθ(1))),
[1 [1 ¬ 1 ¬ 1 [1¬3] 2] 2] has level θ(Ω^(Ωθ(Ω))),
[1 [1 ¬ 1 ¬ 1 [1¬1,2] 2] 2] has level θ(Ω^(Ωθ(Ω^ω))),
[1 [1 ¬ 1 ¬ 1 [1 ¬ 1 \ 2] 2] 2] has level θ(Ω^(Ωθ(Ω^θ(1)))),
[1 [1 ¬ 1 ¬ 1 [1 ¬ 1 [1¬3] 2] 2] 2] has level θ(Ω^(Ωθ(Ω^θ(Ω)))),
[1 [1 ¬ 1 ¬ 1 [1 ¬ 1 ¬ 2] 2] 2] has level θ(Ω^(Ωθ(Ω^Ω))),
[1 [1¬1¬ 1 [1¬1¬ 1 [1¬1¬2] 2] 2] 2] has level θ(Ω^(Ωθ(Ω^(Ωθ(Ω^Ω))))),
[1 [1¬1¬ 1 [1¬1¬ 1 [1¬1¬ 1 [1¬1¬2] 2] 2] 2] 2] has level θ(Ω^(Ωθ(Ω^(Ωθ(Ω^(Ωθ(Ω^Ω))))))).

[1 [1 ¬ 1 ¬ 1 ¬ 2] 2] has level θ(Ω^Ω^2),
[1 [1 ¬ 1 ¬ 1 ¬ 3] 2] has level θ(Ω^((Ω^2)2)),
[1 [1 ¬ 1 ¬ 1 ¬ 1, 2] 2] has level θ(Ω^((Ω^2)ω)),
[1 [1 ¬ 1 ¬ 1 ¬ 1 ¬ 2] 2] has level θ(Ω^Ω^3),
[1 [1 ¬ 1 ¬ 1 ¬ 1 ¬ 1 ¬ 2] 2] has level θ(Ω^Ω^4),
[1 [1 ¬ 1 ¬ 1 ¬ ... ¬ 1 ¬ 2] 2] (with n ¬ symbols) has level θ(Ω^Ω^(n-1)).

In general, when k ≥ 3,
[1 [n_1+1 ¬ n_2+1 ¬ ... ¬ n_k+1] 2] has level θ((Ω^α)ω^n_1),
where α = (Ω^(k-2))n_k + ... + (Ω^2)n_4 + Ωn_3 + n_2.

When encountering separators of [1 [1¬1¬2] 2] or above (i.e. containing hyperseparators of [1¬1¬2] or above, where the ¬ symbol is a 2-hyperseparator), the capital N becomes a chain of arrays separated by ¬ symbols (e.g. N_1 ¬ N_2 ¬ ... ¬ N_k) in the Angle Bracket (A) Rules.

With k ¬ symbols (k ≥ 2),

{a, b [1 [1 ¬ 1 ¬ 1 ¬ ... ¬ 1 ¬ 2] 2] 2} = {a ‹0 [1 ¬ 1 ¬ 1 ¬ ... ¬ 1 ¬ 2] 2› b}
= {a ‹b ‹R_b› b› b},

where R_n = ‘b ¬ b ¬ ... ¬ b ¬ b ‹R_{n-1}› b’ (with k-1 ¬ symbols),
R_1 = ‘0’.

Angle Bracket Rules A5a and A5c are modified as follows:-

Rule A5a (separator [A_{i,p_i}] = [1 ¬ 1 ¬ ... ¬ 1 ¬ 1 [$A_{i+1,1}$] 1 [$A_{i+1,2}$] ... 1 [$A_{i+1,p_{i+1}}$] c_{i+1} $\#_{i+1}$], with d_i ¬ symbols and $d_i \geq 1$, $p_{i+1} \geq 1$):

S_i = ‘b ‹$A_{i,1}$’› b [$A_{i,1}$] b ‹$A_{i,2}$’› b [$A_{i,2}$] ... b ‹A_{i,p_i-1}’› b [A_{i,p_i-1}] b ‹T_i› b [A_{i,p_i}] c_i-1 $\#_i$’,
T_i = ‘b ¬ b ¬ ... ¬ b ¬ S_{i+1}’ (with d_i ¬ symbols).

Increment i by 1 and repeat Rules A5a-d.

Rule A5c (Rules A5a-b do not apply, separator [A_{i,p_i}] = [1 ¬ 1 ¬ 1 ¬ ... ¬ 1 ¬ k #*], with d ¬ symbols and d ≥ 1, k ≥ 2):

S_i = ‘b ‹$A_{i,1}$’› b [$A_{i,1}$] b ‹$A_{i,2}$’› b [$A_{i,2}$] ... b ‹A_{i,p_i-1}’› b [A_{i,p_i-1}] b ‹R_b› b [A_{i,p_i}] c_i-1 $\#_i$’,
R_n = ‘b ¬ b ¬ ... ¬ b ¬ b ‹$A_{i,1}$’› b [$A_{i,1}$] b ‹$A_{i,2}$’› b [$A_{i,2}$] ... b ‹A_{i,p_i-1}’› b [A_{i,p_i-1}] b ‹R_{n-1}› b [A_{i,p_i}] c_i-1 $\#_i$ ¬ k-1 #*’ (with d-1 ¬ symbols prior to ‘b ‹$A_{i,1}$’› b’, n > 1),
R_1 = ‘0’.

(See pages 15-16 of Beyond Bird’s Nested Arrays II for the remainder of the Angle Bracket Rules. The Main (M) Rules always remain the same as in Nested Array Notation.)

In order to extend further, I introduce the second 2-hyperseparator in the series (after ¬). Instead of introducing a double ¬ symbol (e.g. ¬¬ or [2]¬) we let the notation ‘step up a gear’ and introduce a new 3-hyperseparator symbol (a black diamond ♦) to go inside the second 2-hyperseparator as follows: [2♦2]. The [1♦2] 2-hyperseparator ‘drops down’ to the ¬ symbol, just as the [1¬2] hyperseparator ‘drops down’ to the \ symbol. The ♦ symbol requires a minimum of three pairs of square brackets enclosing it in order to allow usage in the ‘base layer’ of a main array (within curly brackets), just as ¬ needs at least two pairs and \ at least one. The [1 [1♦2] 2] hyperseparator ‘drops down’ to [1¬2], which in turn ‘drops down’ to \.

The first separator that requires the new black diamond symbol,

{a, b [1 [1 [2♦2] 2] 2] 2} = {a ‹0 [1 [2♦2] 2] 2› b}
= {a ‹b ‹b ‹1♦2› b› b› b}
= {a ‹b ‹ b ¬ b ¬ ... ¬ b › b› b}
(with b b’s within inner angle brackets).

This is a θ(Ω^Ω^ω)-recursive function.

{a, b [1 [1 [3♦2] 2] 2] 2} = {a ‹0 [1 [3♦2] 2] 2› b}
= {a ‹b ‹b ‹2♦2› b› b› b}
= {a ‹b ‹ b ‹1♦2› b [2♦2] b ‹1♦2› b [2♦2] .... [2♦2] b ‹1♦2› b › b› b}
= {a ‹b ‹ b¬b¬..¬b [2♦2] b¬b¬..¬b [2♦2] .... [2♦2] b¬b¬..¬b › b› b}
(with b^2 b’s and b-1 [2♦2]’s within inner angle brackets).

In general,

{a, b [1 [1 [n+1♦2] 2] 2] 2} = {a ‹0 [1 [n+1♦2] 2] 2› b}

= {a ‹b ‹b ‹n♦2› b› b› b},

where ‘b ‹n♦2› b’ = ‘b ‹n-1♦2› b [n♦2] b ‹n-1♦2› b [n♦2] ... [n♦2] b ‹n-1♦2› b’

(with b ‘b ‹n-1♦2› b’ strings),

‘b ‹1♦2› b’ = ‘b ¬ b ¬ b ¬ ... ¬ b’ (with b b’s),

‘b ‹0♦2› b’ = ‘b’.

Angle Bracket Rules A2 and A3 now read as follows:-

Rule A2 (only 1 entry of either 0 or 1 prior to 2-hyperseparator or higher order hyperseparator):

‘a ‹0 #› b’ = ‘a’,

‘a ‹1 #› b’ = ‘a [1 #] a [1 #] ... [1 #] a’ (with b a’s),

‘a ‹1 ¬ 2› b’ = ‘a \ a \ ... \ a’ (with b a’s),

‘a ‹1 ♦ 2› b’ = ‘a ¬ a ¬ ... ¬ a’ (with b a’s),

where # begins with a 2- or higher order hyperseparator (¬ under no layers of square brackets, or ♦ under less than 2 layers of square brackets).

Rule A3 (last entry in any 1-space or higher dimensional space of array is 1):

‘a ‹# [A] 1› b’ = ‘a ‹#› b’.

When [A] is an m-hyperseparator, [B] is an n-hyperseparator and m < n, or m = n and level of [A] is less than level of [B],

‘a ‹# [A] 1 [B] #*› b’ = ‘a ‹# [B] #*› b’.

Remove trailing 1’s.

The ♦ symbol is a 3-hyperseparator. The ¬ symbol or a separator with a ♦ enclosed by just one square bracket (e.g. [7♦2]) is a 2-hyperseparator. The \ symbol or a separator with a ¬ enclosed by just one square bracket or a ♦ enclosed by two square brackets (e.g. [7¬2] or [7 [7♦2] 2]) is a 1-hyperseparator. A separator with no \, ¬ or ♦ symbols, or all of these enclosed under a minimum of 1, 2 and 3 square brackets respectively, is a normal separator (or 0-hyperseparator).

Rules A5a and A5c are modified as follows:-

Rule A5a ([A_{i,p_i}] = [1 [$B_{i,1}$♦2] 1 [$B_{i,2}$♦2] ... 1 [B_{i,q_i}♦2] 1 [$A_{i+1,1}$] 1 [$A_{i+1,2}$] ... 1 [$A_{i+1,p_{i+1}}$] c_{i+1} $\#_{i+1}$],
where $p_{i+1} \geq 1$, $q_i \geq 1$):

S_i = ‘b ‹$A_{i,1}$’› b [$A_{i,1}$] b ‹$A_{i,2}$’› b [$A_{i,2}$] ... b ‹A_{i,p_i-1}’› b [A_{i,p_i-1}] b ‹T_i› b [A_{i,p_i}] c_i-1 $\#_i$’,

T_i = ‘b ‹$B_{i,1}$’♦2› b [$B_{i,1}$♦2] b ‹$B_{i,2}$’♦2› b [$B_{i,2}$♦2] ... b ‹B_{i,q_i}’♦2› b [B_{i,q_i}♦2] S_{i+1}’.

Increment i by 1 and repeat Rules A5a-d.

Rule A5c (Rules A5a-b do not apply, [A_{i,p_i}] = [1 [$B_{i,1}$♦2] 1 [$B_{i,2}$♦2] ... 1 [B_{i,q_i-1}♦2] 1 ¬ d_i $\#^*_i$],
where $q_i \geq 1$):

S_i = ‘b ‹$A_{i,1}$’› b [$A_{i,1}$] b ‹$A_{i,2}$’› b [$A_{i,2}$] ... b ‹A_{i,p_i-1}’› b [A_{i,p_i-1}] b ‹R_b› b [A_{i,p_i}] c_i-1 $\#_i$’,

R_n = ‘b ‹$B_{i,1}$’♦2› b [$B_{i,1}$♦2] b ‹$B_{i,2}$’♦2› b [$B_{i,2}$♦2] ... b ‹B_{i,q_i-1}’♦2› b [B_{i,q_i-1}♦2]
b ‹$A_{i,1}$’› b [$A_{i,1}$] b ‹$A_{i,2}$’› b [$A_{i,2}$] ... b ‹A_{i,p_i-1}’› b [A_{i,p_i-1}] b ‹R_{n-1}› b [A_{i,p_i}] c_i-1 $\#_i$
¬ d_i-1 $\#^*_i$’ (n > 1),

R_1 = ‘0’.

Like each $A_{i,j}$, each $B_{i,j}$ is a string of characters within a separator. The main difference between the A strings and the B strings is that ♦2 is appended to each of the latter (in order to turn them into

2-hyperseparators) as their rank is either equal to or greater than that of the ¬ symbol (or [1♦2]), whereas each of the [$A_{i,j}$] is below ¬ in rank. If B_{i,q_i-1} = '1', the separator [B_{i,q_i-1}♦2] would be the ¬ symbol and 'b ‹B_{i,q_i-1}'♦2› b' would become 'b ‹0♦2› b' = 'b'.

The old Rule A5d becomes Rule A5e (Rules A5a-d do not apply) and a new Rule A5d is created as follows:-

Rule A5d (Rules A5a-c do not apply, [A_{i,p_i}] = [1 [$B_{i,1}$♦2] 1 [$B_{i,2}$♦2] ... 1 [B_{i,q_i}♦2] d_i $\#*_i$],
where $q_i \geq 1$ and [B_{i,q_i}♦2] is above [1♦2] or ¬ in level):

S_i = 'b ‹$A_{i,1}$'› b [$A_{i,1}$] b ‹$A_{i,2}$'› b [$A_{i,2}$] ... b ‹A_{i,p_i-1}'› b [A_{i,p_i-1}] b ‹T› b [A_{i,p_i}] c_i-1 $\#_i$',

T = 'b ‹$B_{i,1}$'♦2› b [$B_{i,1}$♦2] b ‹$B_{i,2}$'♦2› b [$B_{i,2}$♦2] ... b ‹B_{i,q_i}'♦2› b [B_{i,q_i}♦2] d_i-1 $\#*_i$'.

The new Rule A5d is necessary owing to the fact that if the old Rule A5d (now A5e) applied, the string T would become

A_{i,p_i}' = '0 [$B_{i,1}$♦2] 1 [$B_{i,2}$♦2] ... 1 [B_{i,q_i}♦2] d_i $\#*_i$',

which means that

'b ‹T› b' = 'b ‹ 0 [$B_{i,1}$♦2] 1 [$B_{i,2}$♦2] ... 1 [B_{i,q_i}♦2] d_i $\#*_i$ › b'
= 'b',

by the revised Rule A2, since [$B_{i,1}$♦2] is a 2-hyperseparator (start of the $\#_0$ string).

Take the simple example of

N = {a, b [1 [1 [2♦2] 2] 2] 2}
= {a ‹0 [1 [2♦2] 2] 2› b},

with $c_1 = 2$, $d_1 = 2$, $p_1 = 1$, $q_1 = 1$, all #-strings blank,
[$A_{1,1}$] = [1 [2♦2] 2],
[$B_{1,1}$♦2] = [2♦2].

Under the old Rule A5d,

N = {a ‹S› b}
= {a ‹b ‹T› b› b}
= {a ‹b› b},

whereas under the new Rule A5d,

N = {a ‹S› b}
= {a ‹b ‹T› b› b}
= {a ‹b ‹b ‹$B_{1,1}$'♦2› b› b› b}
= {a ‹b ‹b ‹1♦2› b› b› b},

and, by the revised Rule A2,

N = {a ‹b ‹ b ¬ b ¬ ... ¬ b › b› b} (with b b's within inner angle brackets),

as originally defined.

Just as Rule A5a processes branching separators, and A5b and A5c deal with nesting separators, the new A5d is the subrule for what I term plugging separators, as while they are virtually identical to nesting separators, plugging separators merely 'plug' an angle bracket array with only one nested layer in the space to the left of them, rather than 'nest' a number of layers of the angle bracket array. The separator

[A_{i,p_i}] = [1 [$B_{i,1}$♦2] 1 [$B_{i,2}$♦2] ... 1 [B_{i,q_i}♦2] d_i $\#*_i$]

triggers Rule A5c if B_{i,q_i} = '1' but A5d otherwise.

The initial part of Rule A5 now reads as follows:-

Rule A5 (Rules A1-4 do not apply, first entry is 0, separator immediately prior to next non-1 entry (c_1) is [A_{1,p_1}]):

‘a ‹ 0 [$A_{1,1}$] 1 [$A_{1,2}$] ... 1 [A_{1,p_1}] c_1 $\#_1$ $\#_0$ › b’ = ‘a ‹S_1 $\#_0$› b’,

where $p_1 \geq 1$, $\#_1$ contains no 2- or higher order hyperseparators in its base layer and $\#_0$ is either an empty string or begins with a 2- or higher order hyperseparator (¬ under no layers of square brackets, or ♦ under less than 2 layers of square brackets).
Set i = 1 and follow Rules A5a-e (b-e are terminal, a is not).

The final line of Rule A5a now reads: Increment i by 1 and repeat Rules A5a-e.

The diamond symbol allows us to introduce the following separators:-

[1 [1 [2♦2] 2] 2] has level θ(Ω^Ω^ω),
[1 [1 [2♦2] 2] 1 \ 2] has level θ(Ω^Ω^ω+1),
[1 [1 [2♦2] 2] 1 [2¬2] 2] has level θ(Ω^Ω^ω+ω),
[1 [1 [2♦2] 2] 1 [1¬3] 2] has level θ(Ω^Ω^ω+Ω),
[1 [1 [2♦2] 2] 1 [1¬1¬2] 2] has level θ(Ω^Ω^ω+Ω^Ω),
[1 [1 [2♦2] 2] 1 [1 [2♦2] 2] 2] has level θ((Ω^Ω^ω)2),
[1 [2 [2♦2] 2] 2] has level θ((Ω^Ω^ω)ω),
[1 [1 \ 2 [2♦2] 2] 2] has level θ((Ω^Ω^ω)θ(1)),
[1 [1 [1¬3] 2 [2♦2] 2] 2] has level θ((Ω^Ω^ω)θ(Ω)),
[1 [1 [1¬1¬2] 2 [2♦2] 2] 2] has level θ((Ω^Ω^ω)θ(Ω^Ω)),
[1 [1 [1 [2♦2] 2] 2 [2♦2] 2] 2] has level θ((Ω^Ω^ω)θ(Ω^Ω^ω)),
[1 [1 [1 [1 [2♦2] 2] 2 [2♦2] 2] 2 [2♦2] 2] 2] has level θ((Ω^Ω^ω)θ((Ω^Ω^ω)θ(Ω^Ω^ω))),
[1 [1 ¬ 2 [2♦2] 2] 2] has level θ(Ω^(Ω^ω+1)),
[1 [1 ¬ 1 ¬ 2 [2♦2] 2] 2] has level θ(Ω^(Ω^ω+Ω)),
[1 [1 ¬ 1 ¬ 1 ¬ 2 [2♦2] 2] 2] has level θ(Ω^(Ω^ω+Ω^2)),
[1 [1 [2♦2] 3] 2] has level θ(Ω^((Ω^ω)2)),
[1 [1 [2♦2] 1 \ 2] 2] has level θ(Ω^((Ω^ω)θ(1))),
[1 [1 [2♦2] 1 [1¬3] 2] 2] has level θ(Ω^((Ω^ω)θ(Ω))),
[1 [1 [2♦2] 1 [1¬1¬2] 2] 2] has level θ(Ω^((Ω^ω)θ(Ω^Ω))),
[1 [1 [2♦2] 1 [1 [2♦2] 2] 2] 2] has level θ(Ω^((Ω^ω)θ(Ω^Ω^ω))),
[1 [1 [2♦2] 1 [1 [2♦2] 1 [1 [2♦2] 2] 2] 2] 2] has level θ(Ω^((Ω^ω)θ(Ω^((Ω^ω)θ(Ω^Ω^ω))))).

[1 [1 [2♦2] 1 ¬ 2] 2] has level θ(Ω^Ω^(ω+1)),
[1 [1 [2♦2] 1 ¬ 1 ¬ 2] 2] has level θ(Ω^Ω^(ω+2)),
[1 [1 [2♦2] 1 [2♦2] 2] 2] has level θ(Ω^Ω^(ω2)),
[1 [1 [2♦2] 1 [2♦2] 1 [2♦2] 2] 2] has level θ(Ω^Ω^(ω3)),
[1 [1 [3♦2] 2] 2] has level θ(Ω^Ω^ω^2),
[1 [1 [4♦2] 2] 2] has level θ(Ω^Ω^ω^3),
[1 [1 [1,2 ♦2] 2] 2] has level θ(Ω^Ω^ω^ω),
[1 [1 [1 \ 2 ♦2] 2] 2] has level θ(Ω^Ω^θ(1)),
[1 [1 [1 [1¬3] 2 ♦2] 2] 2] has level θ(Ω^Ω^θ(Ω)),
[1 [1 [1 [1¬1,2] 2 ♦2] 2] 2] has level θ(Ω^Ω^θ(Ω^ω)),
[1 [1 [1 [1¬1¬2] 2 ♦2] 2] 2] has level θ(Ω^Ω^θ(Ω^Ω)),
[1 [1 [1 [1 [2♦2] 2] 2 ♦2] 2] 2] has level θ(Ω^Ω^θ(Ω^Ω^ω)),
[1 [1 [1 [1 [1 \ 2 ♦2] 2] 2 ♦2] 2] 2] has level θ(Ω^Ω^θ(Ω^Ω^θ(1))),

[1 [1 [1 [1 [1 [1 [1 \ 2 ♦2] 2] 2 ♦2] 2] 2 ♦2] 2] 2] has level θ(Ω^Ω^θ(Ω^Ω^θ(Ω^Ω^θ(1)))).

If [X] and [Y] are 2-hyperseparators (¬ or arrays separated by ♦ symbols) such that [1 [1 [X] 2] 2] and [1 [1 [Y] 2] 2] have levels of θ(Ω^Ω^α) and θ(Ω^Ω^β) respectively (where α ≥ β), then
[1 [1 [X] 1 [Y] 2] 2] would have level θ(Ω^Ω^(α+β)).

[1 [1 [1♦3] 2] 2] has level θ(Ω^Ω^Ω),
[1 [1 [1♦3] 1 ¬ 2] 2] has level θ(Ω^Ω^(Ω+1)),
[1 [1 [1♦3] 1 [2♦2] 2] 2] has level θ(Ω^Ω^(Ω+ω)),
[1 [1 [1♦3] 1 [1 \ 2 ♦2] 2] 2] has level θ(Ω^Ω^(Ω+θ(1))),
[1 [1 [1♦3] 1 [1 [1 [1♦3] 2] 2 ♦2] 2] 2] has level θ(Ω^Ω^(Ω+θ(Ω^Ω^Ω))),
[1 [1 [1♦3] 1 [1 [1 [1♦3] 1 [1 [1 [1♦3] 2] 2 ♦2] 2] 2 ♦2] 2] 2] has level
θ(Ω^Ω^(Ω+θ(Ω^Ω^(Ω+θ(Ω^Ω^Ω))))),
[1 [1 [1♦3] 1 [1♦3] 2] 2] has level θ(Ω^Ω^(Ω2)),
[1 [1 [2♦3] 2] 2] has level θ(Ω^Ω^(Ωω)),
[1 [1 [1 \ 2 ♦3] 2] 2] has level θ(Ω^Ω^(Ωθ(1))),
[1 [1 [1 [1 [1♦3] 2] 2 ♦3] 2] 2] has level θ(Ω^Ω^(Ωθ(Ω^Ω^Ω))),
[1 [1 [1 [1 [1 [1 [1♦3] 2] 2 ♦3] 2] 2 ♦3] 2] 2] has level θ(Ω^Ω^(Ωθ(Ω^Ω^(Ωθ(Ω^Ω^Ω))))).

[1 [1 [1♦4] 2] 2] has level θ(Ω^Ω^Ω^2),
[1 [1 [1♦5] 2] 2] has level θ(Ω^Ω^Ω^3),
[1 [1 [1♦1,2] 2] 2] has level θ(Ω^Ω^Ω^ω),
[1 [1 [1 ♦ 1 \ 2] 2] 2] has level θ(Ω^Ω^Ω^θ(1)),
[1 [1 [1 ♦ 1 [1 ¬ 3] 2] 2] 2] has level θ(Ω^Ω^Ω^θ(Ω)),
[1 [1 [1 ♦ 1 [1 ¬ 1 ¬ 2] 2] 2] 2] has level θ(Ω^Ω^Ω^θ(Ω^Ω)),
[1 [1 [1 ♦ 1 [1 [1♦3] 2] 2] 2] 2] has level θ(Ω^Ω^Ω^θ(Ω^Ω^Ω)),
[1 [1 [1 ♦ 1 [1 [1 ♦ 1 \ 2] 2] 2] 2] 2] has level θ(Ω^Ω^Ω^θ(Ω^Ω^Ω^θ(1))),
[1 [1 [1 ♦ 1 [1 [1 ♦ 1 [1 [1 ♦ 1 \ 2] 2] 2] 2] 2] 2] 2] has level
θ(Ω^Ω^Ω^θ(Ω^Ω^Ω^θ(Ω^Ω^Ω^θ(1)))),
[1 [1 [1 ♦ 1 ♦ 2] 2] 2] has level θ(Ω^Ω^Ω^Ω).

Notice some similarities of the 10 separators above with the 10 below:
[1 [1¬4] 2] has level θ(Ω^2),
[1 [1¬5] 2] has level θ(Ω^3),
[1 [1¬1,2] 2] has level θ(Ω^ω),
[1 [1 ¬ 1 \ 2] 2] has level θ(Ω^θ(1)),
[1 [1 ¬ 1 \ 3] 2] has level θ(Ω^θ(1, 1)),
[1 [1 ¬ 1 \ 1 \ 2] 2] has level θ(Ω^θ(2)),
[1 [1 ¬ 1 [1 ¬ 3] 2] 2] has level θ(Ω^θ(Ω)),
[1 [1 ¬ 1 [1 ¬ 1 \ 2] 2] 2] has level θ(Ω^θ(Ω^θ(1))),
[1 [1 ¬ 1 [1 ¬ 1 [1 ¬ 1 \ 2] 2] 2] 2] has level θ(Ω^θ(Ω^θ(Ω^θ(1)))),
[1 [1 ¬ 1 ¬ 2] 2] has level θ(Ω^Ω).

The creation of the first group of 10 separators mirrors the second group of 10, except that the ¬'s are ♦'s, the single nested square brackets become double nested square brackets and the single Ω's in the θ function become Ω^Ω^Ω's. Another pattern we are spotting is that when X and Y are arrays such that [1 [X¬Y] 2] has level θ(α), then [1 [1 [X♦Y] 2] 2] would have level θ(Ω^Ω^α).

It is not possible to place the 3-hyperseparator ♦ symbol on the same 'nested level' as a 2-hyperseparator as in this example: [1 [1 [1 ¬ 3 ♦2] 2] 2]. The '1¬3' could become '1¬2' which 'drops

down' to a backslash (\) as in [1 [1 [\ ♦2] 2] 2] and each opening bracket must be followed by a number. Nor are [1 [1 [1 [1♦3] 2 ♦2] 2] 2] and [1 [1 [1 ♦ 1 ¬ 3] 2] 2] allowed ([1 [1 [1 ♦ 1 ¬ 2] 2] 2] 'drops down' to [1 [1 [1 ♦ \] 2] 2] and each closing bracket must follow a number).
[1 [1 [1 ♦ 1 [1♦3] 2] 2] 2] is also illegal.

The θ(Ω^Ω^Ω) level separator

{a, b [1 [1 [1♦3] 2] 2] 2} = {a ‹0 [1 [1♦3] 2] 2› b}
= {a ‹b ‹b ‹b ‹b ‹b ‹ ... ‹b ‹b ‹b ‹b♦2› b› b♦2› b› ... › b♦2› b› b♦2› b› b› b}
(with 2b b's from centre to right and b-1 ♦'s)
= {a ‹b ‹R_b› b› b},

where R_n = 'b ‹b ‹R_{n-1}› b ♦2› b',
R_1 = '0'.

For example, when a = 3 and b = 2,

{3, 2 [1 [1 [1♦3] 2] 2] 2} = {3 ‹0 [1 [1♦3] 2] 2› 2}
= {3 ‹2 ‹2 ‹2♦2› 2› 2› 2}
= {3 ‹2 ‹ 2¬2 [2♦2] 2¬2 › 2› 2}
= {3 ‹ 2 [1¬2 [2♦2] 2¬2] 2 [2¬2 [2♦2] 2¬2] 2 [1¬2 [2♦2] 2¬2] 2 › 2}
(since '2 ‹0¬2 [2♦2] 2¬2› 2' = '2')
= {S [1 [1¬2 [2♦2] 2¬2] 2 [2¬2 [2♦2] 2¬2] 2 [1¬2 [2♦2] 2¬2] 2]
S [2 [1¬2 [2♦2] 2¬2] 2 [2¬2 [2♦2] 2¬2] 2 [1¬2 [2♦2] 2¬2] 2]
S [1 [1¬2 [2♦2] 2¬2] 2 [2¬2 [2♦2] 2¬2] 2 [1¬2 [2♦2] 2¬2] 2] S},

where S = '3 ‹ 0 [1¬2 [2♦2] 2¬2] 2 [2¬2 [2♦2] 2¬2] 2 [1¬2 [2♦2] 2¬2] 2 › 2'.
By Rule A5c with

$c_1 = 2$, $d_1 = 2$, $p_1 = 1$, $q_1 = 1$,
[$A_{1,1}$] = [1¬2 [2♦2] 2¬2],
[$B_{1,1}$♦2] = [1♦2] = ¬,
$\#_1$ = '[2¬2 [2♦2] 2¬2] 2 [1¬2 [2♦2] 2¬2] 2',
$\#^*_1$ = '[2♦2] 2¬2',

we obtain

S = '3 ‹S_1› 2' (initial part of Rule A5),
S_1 = '2 ‹R_2› 2 [1¬2 [2♦2] 2¬2] 1 [2¬2 [2♦2] 2¬2] 2 [1¬2 [2♦2] 2¬2] 2',
R_2 = '2 [1¬2 [2♦2] 2¬2] 1 [2¬2 [2♦2] 2¬2] 2 [1¬2 [2♦2] 2¬2] 2 ¬ 1 [2♦2] 2 ¬ 2',

since '2 ‹R_1› 2' = '2 ‹0› 2' = '2'.
It would be rather tedious to go any further – we would deal with '2 ‹R_2› 2' next, and repeat Rule A5c for

'2 ‹ 0 [1¬2 [2♦2] 2¬2] 1 [2¬2 [2♦2] 2¬2] 2 [1¬2 [2♦2] 2¬2] 2 ¬ 1 [2♦2] 2 ¬ 2 › 2'.

Simply increasing the value of b to 3 and 4 gives

{3, 3 [1 [1 [1♦3] 2] 2] 2} = {3 ‹0 [1 [1♦3] 2] 2› 3}
= {3 ‹3 ‹3 ‹3 ‹3 ‹3♦2› 3› 3♦2› 3› 3› 3},
{3, 4 [1 [1 [1♦3] 2] 2] 2} = {3 ‹0 [1 [1♦3] 2] 2› 4}
= {3 ‹4 ‹4 ‹4 ‹4 ‹4 ‹4 ‹4♦2› 4› 4♦2› 4› 4♦2› 4› 4› 4}.

We see the creation of a Nested Hyper-Nested Array Notation. The [1 [1 [1♦3] 2] 2] separator marks the beginning of the second level of this notation, since the nesting separator [1 [1♦3] 2] is the lowest special hyperseparator to require two pairs of brackets per iteration of the R_n function, and thus can be regarded as a 2-nesting separator. The [1 [1¬3] 2] separator can be said to mark the beginning of the first level of this notation, since the second lowest nesting separator [1¬3] has a slightly different rule to that of the lowest nesting separator (\). By this token, the separators [1 \ 1 \ 2], [1 [1¬1¬2] 2] and [1 [1 [1♦1♦2] 2] 2] mark the midway points (or 'half levels') at levels 0, 1 and 2 respectively.

The θ(Ω^Ω^Ω^(k-2)) level separator

{a, b [1 [1 [1♦k] 2] 2] 2} = {a ‹0 [1 [1♦k] 2] 2› b}
= {a ‹b ‹R_b› b› b},

where R_n = 'b ‹b ‹R_{n-1}› b ♦k-1› b',
R_1 = '0'.

The θ(Ω^Ω^Ω^ω) level separator

{a, b [1 [1 [1♦1,2] 2] 2] 2} = {a ‹0 [1 [1♦1,2] 2] 2› b}
= {a ‹b ‹b ‹b♦b› b› b› b},

which bears similarities to

{a, b [1 [1¬1,2] 2] 2} = {a ‹0 [1¬1,2] 2› b}
= {a ‹b ‹b¬b› b› b}

and {a, b [1 \ 1, 2] 2} = {a ‹0 \ 1, 2› b}
= {a ‹b \ b› b}.

The initial equation of Rule A5 is

'a ‹ 0 [$A_{1,1}$] 1 [$A_{1,2}$] ... 1 [A_{1,p_1}] c_1 $\#_1$ $\#_0$ › b' = 'a ‹S_1 $\#_0$› b',

where each of the [$A_{1,j}$] are either normal separators (each ¬ within at least 2 layers of square brackets, each ♦ within at least 3 layers) or 1-hyperseparators (each ♦ within at least 2 layers of square brackets), $p_1 \geq 1$, $\#_1$ is the remainder of the angle bracket array that contains no 2- or higher order hyperseparators (including bare ¬ or ♦ signs, not 'covered' by any layers of square brackets) and $\#_0$ is the remainder of the angle bracket array beginning with the first 2- or higher order hyperseparator, or empty (null string) if there are no such hyperseparators. It is the [A_{1,p_1}] separator which we are concerned with next.

Nesting separators (triggers Rules A5b or A5c) include the backslash (\) and hyperseparators of the form

[1 [B_1] 1 [B_2] ... 1 [B_q] d #*],

where each of the [B_j] are 2-hyperseparators (either ¬ sign or of the form [X♦Y], where X and Y are arrays), [B_q] is a nesting 2-hyperseparator (either ¬ sign or of the form [1 ♦ k #**]), q ≥ 1, d ≥ 2 and #* is rest of array.

If [B_q] above is of the form [X♦Y] instead, where X and Y are arrays other than the '1' string (i.e. X ≠ '1' and Y ≠ '1'), then

[1 [B_1] 1 [B_2] ... 1 [B_q] d #*]

would be a plugging separator and [B_q] a plugging 2-hyperseparator.

If [B_q] above is of the form

[1 ♦ 1 [A_1] 1 [A_2] ... 1 [A_p] k #**]

instead, where each of the [A_j] are either normal or 1-hyperseparators and p ≥ 1, then

[1 [B_1] 1 [B_2] ... 1 [B_q] d #*]

would be a branching separator and [B_q] a branching 2-hyperseparator (investigated later).

If [A_{1,p_1}] is a backslash (\), Rule A5b is executed. If [A_{1,p_1}] is any other nesting separator, A5c is utilised. Rule A5c splits into two subrules – A5c1 ([B_q] is ¬ sign) and A5c2 ([B_q] = [1 ♦ k #**], other nesting 2-hyperseparator). The latter subrule entails two pairs of brackets per iteration of the R_n function, with the ♦ symbol on a different nested level to the [B_j] 2-hyperseparators.

Rule A5c1 is basically similar to the previous Rule A5c (the i-subscripts have been removed):

S = ‘b ‹A_1’› b [A_1] b ‹A_2’› b [A_2] ... b ‹A_{p-1}’› b [A_{p-1}] b ‹R_b› b [A_p] c-1 #’,
R_n = ‘b ‹B_1’› b [B_1] b ‹B_2’› b [B_2] ... b ‹B_{q-1}’› b [B_{q-1}]
b ‹A_1’› b [A_1] b ‹A_2’› b [A_2] ... b ‹A_{p-1}’› b [A_{p-1}] b ‹R_{n-1}› b [A_p] c-1 # ¬ d-1 #*’ (n > 1),
R_1 = ‘0’.

Rule A5c2 ([B_q] = [1 ♦ k #**], where k ≥ 2 and #** is rest of array):

S = ‘b ‹A_1’› b [A_1] b ‹A_2’› b [A_2] ... b ‹A_{p-1}’› b [A_{p-1}] b ‹R_b› b [A_p] c-1 #’,
R_n = ‘b ‹B_1’› b [B_1] b ‹B_2’› b [B_2] ... b ‹B_{q-1}’› b [B_{q-1}] b ‹T_n› b [B_q] d-1 #*’ (n > 1),
T_n = ‘b ‹A_1’› b [A_1] b ‹A_2’› b [A_2] ... b ‹A_{p-1}’› b [A_{p-1}] b ‹R_{n-1}› b [A_p] c-1 # ♦ k-1 #**’ (n > 1),
R_1 = ‘0’.

In short, Rule A5c1 processes 1-nesting separators and A5c2 deals with 2-nesting separators.

Here are two examples of applications of Rule A5c2 to rather similar looking arrays. The second example is the larger of the two.

Example 1:

N = {a, b [1 [1 [1♦4] 2] 1 [1 [1♦3] d] c] 2}
= {a ‹0 [1 [1♦4] 2] 1 [1 [1♦3] d] c› b},

with k = 3, p = 2, q = 1, all #-strings blank,
[A_1] = [1 [1♦4] 2],
[A_2] = [1 [1♦3] d],
[B_1] = [1♦3].

Since the string to the left of [1 [1♦4] 2] is ‘b ‹0 [1♦4] 2› b’ = ‘b’ (as [1♦4] is a 2-hyperseparator and thus the beginning of an $\#_0$ string in the revised Rule A2), it follows that

N = {a ‹ b [1 [1♦4] 2] b ‹R_b› b [1 [1♦3] d] c-1 › b},

where R_n = ‘b ‹ b [1 [1♦4] 2] b ‹R_{n-1}› b [1 [1♦3] d] c-1 ♦2 › b [1♦3] d-1’,
R_1 = ‘0’.

Example 2:

N = {a, b [1 [1 [1♦4] 1 [1♦3] d] c] 2}
= {a ‹0 [1 [1♦4] 1 [1♦3] d] c› b},

with k = 3, p = 1, q = 2, all #-strings blank,
[A_1] = [1 [1♦4] 1 [1♦3] d],
[B_1] = [1♦4],
[B_2] = [1♦3].

Since the string to the left of [1♦4] is ‘b ‹0♦4› b’ = ‘b’ (as ♦ is a 3-hyperseparator and thus the beginning of an $\#_0$ string in the revised Rule A2), it follows that

N = {a ‹ b ‹R_b› b [1 [1♦4] 1 [1♦3] d] c-1 › b},

where R_n = ‘b [1♦4] b ‹ b ‹R_{n-1}› b [1 [1♦4] 1 [1♦3] d] c-1 ♦2 › b [1♦3] d-1’,
R_1 = ‘0’.

When a = 3, b = 3, c = 2 and d = 2, the above two examples are

{3 ‹3 [1 [1♦4] 2] 3 ‹3 ‹3 [1 [1♦4] 2] 3 ‹3 ‹3 [1 [1♦4] 2] 3 ♦2› 3› 3 ♦2› 3› 3› 3}

and {3 ‹3 ‹3 [1♦4] 3 ‹3 ‹3 [1♦4] 3 ‹3♦2› 3› 3 ♦2› 3› 3› 3} respectively, and the separators [1 [1 [1♦4] 2] 1 [1 [1♦3] 2] 2] and [1 [1 [1♦4] 1 [1♦3] 2] 2] have levels of θ(Ω^Ω^Ω^2+Ω^Ω^Ω) and θ(Ω^Ω^(Ω^2+Ω)) respectively.

In a rather more complicated example,

N = {4, 3 [1 [1 [1♦5] 2] 1 [1 [3♦4] 1 [1♦4] 3] 6 [5♦2] 2] 2}
= {4 ‹ 0 [1 [1♦5] 2] 1 [1 [3♦4] 1 [1♦4] 3] 6 [5♦2] 2 › 3}

would involve an application of Rule A5c2 with

a = 4, b = 3, c = 6, d = 3, k = 4, p = 2, q = 2,
[A_1] = [1 [1♦5] 2],
[A_2] = [1 [3♦4] 1 [1♦4] 3],
[B_1] = [3♦4],
[B_2] = [1♦4],
$\#_0$ = '[5♦2] 2' (as it begins with a 2-hyperseparator),
#, #*, #** are null strings.

Since the string to the left of [1 [1♦5] 2] is '3 ‹0 [1♦5] 2› 3' = '3' (as [1♦5] is a 2-hyperseparator and thus the beginning of an $\#_0$ string in the revised Rule A2), it follows that

N = {4 ‹ S [5♦2] 2 › 3},

where S = '3 [1 [1♦5] 2] 3 ‹R_3› 3 [1 [3♦4] 1 [1♦4] 3] 5',
R_3 = '3 ‹2♦4› 3 [3♦4] 3 ‹T_3› 3 [1♦4] 2',
T_3 = '3 [1 [1♦5] 2] 3 ‹R_2› 3 [1 [3♦4] 1 [1♦4] 3] 5 ♦3',
R_2 = '3 ‹2♦4› 3 [3♦4] 3 ‹T_2› 3 [1♦4] 2',
T_2 = '3 [1 [1♦5] 2] 3 [1 [3♦4] 1 [1♦4] 3] 5 ♦3',

since '3 ‹R_1› 3' = '3 ‹0› 3' = '3'.

By Rule A6,

'3 ‹2♦4› 3' = '3 [1♦4] 3 [1♦4] 3 [2♦4] 3 [1♦4] 3 [1♦4] 3 [2♦4] 3 [1♦4] 3 [1♦4] 3',

since '3 ‹0♦4› 3' = '3'
(as ♦ is a 2-hyperseparator and the start of $\#_0$ in the revised Rule A2).

Plugging separators (triggers Rule A5d) include hyperseparators of the form

[1 [B_1] 1 [B_2] ... 1 [B_q] d #*], where [B_q] = [X♦Y], X ≠ '1', Y ≠ '1',

where each of the [B_j] are 2-hyperseparators, q ≥ 1, d ≥ 2 and #* is rest of array.

Rule A5d reads as follows (the i-subscripts have been removed as in Rules A5c1-2):

S = 'b ‹A_1'› b [A_1] b ‹A_2'› b [A_2] ... b ‹A_{p-1}'› b [A_{p-1}] b ‹T› b [A_p] c-1 #',
T = 'b ‹B_1'› b [B_1] b ‹B_2'› b [B_2] ... b ‹B_q'› b [B_q] d-1 #*'.

Branching separators (triggers Rule A5a) include hyperseparators of the form

[1 [B_1] 1 [B_2] ... 1 [B_q] 1 [A_1] 1 [A_2] ... 1 [A_p] k #*]

and [1 [B_1] 1 [B_2] ... 1 [B_q] d #*], where [B_q] = [1 ♦ 1 [A_1] 1 [A_2] ... 1 [A_p] k #**],

where each of the [B_j] are 2-hyperseparators, each of the [A_j] are either normal separators or 1-hyperseparators, p ≥ 1, q ≥ 1, d ≥ 2, k ≥ 2 and #* and #** are rest of their respective arrays. Rule A5a also splits into two subrules – A5a1 (for 1-branching separators) and A5a2 (for 2-branching separators). Note that i is initially set to 1.

Rule A5a1 ([A_{i,p_i}] = [1 [$B_{i,1}$] 1 [$B_{i,2}$] ... 1 [B_{i,q_i}] 1 [$A_{i+1,1}$] 1 [$A_{i+1,2}$] ... 1 [$A_{i+1,p_{i+1}}$] c_{i+1} $\#_{i+1}$],
where $p_{i+1} \geq 1$, $q_i \geq 1$):

S_i = 'b ‹$A_{i,1}$'› b [$A_{i,1}$] b ‹$A_{i,2}$'› b [$A_{i,2}$] ... b ‹A_{i,p_i-1}'› b [A_{i,p_i-1}] b ‹T_i› b [A_{i,p_i}] c_i-1 $\#_i$',
T_i = 'b ‹$B_{i,1}$'› b [$B_{i,1}$] b ‹$B_{i,2}$'› b [$B_{i,2}$] ... b ‹B_{i,q_i}'› b [B_{i,q_i}] S_{i+1}'.

Increment i by 1 and repeat Rules A5a-e.

Rule A5a2 ($[A_{i,p_i}]$ = [1 [$B_{i,1}$] 1 [$B_{i,2}$] ... 1 [B_{i,q_i}] d_i $\#^*_i$],
where [B_{i,q_i}] = [1 ♦ 1 [$A_{i+1,1}$] 1 [$A_{i+1,2}$] ... 1 [$A_{i+1,p_{i+1}}$] c_{i+1} $\#_{i+1}$]
and $p_{i+1} \ge 1$, $q_i \ge 1$):

S_i = ‘b ‹$A_{i,1}$’› b [$A_{i,1}$] b ‹$A_{i,2}$’› b [$A_{i,2}$] ... b ‹A_{i,p_i-1}’› b [A_{i,p_i-1}] b ‹T_i› b [A_{i,p_i}] c_i-1 $\#_i$’,

T_i = ‘b ‹$B_{i,1}$’› b [$B_{i,1}$] b ‹$B_{i,2}$’› b [$B_{i,2}$] ... b ‹b ♦ S_{i+1}› b [B_{i,q_i}] d_i-1 $\#^*_i$’.

Increment i by 1 and repeat Rules A5a-e.

For example,

N = {a, b [1 [1 [1 ♦ 1 [1 [1♦1,2] 2] 2] 2] 2] 2}
= {a ‹0 [1 [1 ♦ 1 [1 [1♦1,2] 2] 2] 2] 2› b}

would involve a double application of Rule A5a2 with

c_i = 2, d_i = 2, p_i = 1, q_i = 1,
[$A_{1,1}$] = [1 [$B_{1,1}$] 2],
[$B_{1,1}$] = [1 ♦ 1 [$A_{2,1}$] 2],
[$A_{2,1}$] = [1 [$B_{2,1}$] 2],
[$B_{2,1}$] = [1 ♦ 1 [$A_{3,1}$] 2],
[$A_{3,1}$] = [1] (comma),
$\#_0$, $\#_i$, $\#^*_i$ are null strings.

It follows that,

N = {a ‹S_1› b},

where S_1 = ‘b ‹T_1› b’,
T_1 = ‘b ‹b ♦ S_2› b’,
S_2 = ‘b ‹T_2› b’,
T_2 = ‘b ‹b ♦ S_3› b’.

Putting this together gives,

N = {a ‹b ‹b ‹b ♦ b ‹b ‹b♦S_3› b› b› b› b› b}.

Finally, we finish by using Rule A5d (S_3 = ‘b ‹$A_{3,1}$’› b’ = ‘b ‹0› b’ = ‘b’), which yields

N = {a ‹b ‹b ‹b ♦ b ‹b ‹b♦b› b› b› b› b› b}.

The lowest separator with more than one ♦ symbol,

{a, b [1 [1 [1♦1♦2] 2] 2] 2} = {a ‹0 [1 [1♦1♦2] 2] 2› b}
= {a ‹b ‹R_b› b› b},

where R_n = ‘b ‹b ♦ b ‹R_{n-1}› b› b’,
R_1 = ‘0’.

This is equivalent to the array N above in which are b-1 double layers instead of just two, or

{a, b [1 [1 [1 ♦ 1 [1 [1 ♦ 1 [... [1 [1 ♦ 1 [1 [1♦1,2] 2] 2] 2] ...] 2] 2] 2] 2] 2] 2}
(with 2b-1 square brackets and b-1 ♦ symbols, for b ≥ 2)

and is a θ(Ω^Ω^Ω^Ω)-recursive function.

We now extend the separators to multiple diamonds on the same ‘nested level’.

[1 [1 [1♦1♦2] 2] 2] has level θ(Ω^Ω^Ω^Ω),
[1 [1 [1♦1♦2] 2] 1 [1 [1♦1♦2] 2] 2] has level θ((Ω^Ω^Ω^Ω)2),
[1 [2 [1♦1♦2] 2] 2] has level θ((Ω^Ω^Ω^Ω)ω),
[1 [1 \ 2 [1♦1♦2] 2] 2] has level θ((Ω^Ω^Ω^Ω)θ(1)),
[1 [1 [1 [1♦1♦2] 2] 2 [1♦1♦2] 2] 2] has level θ((Ω^Ω^Ω^Ω)θ(Ω^Ω^Ω^Ω)),
[1 [1 [1 [1 [1♦1♦2] 2] 2 [1♦1♦2] 2] 2 [1♦1♦2] 2] 2] has level
θ((Ω^Ω^Ω^Ω)θ((Ω^Ω^Ω^Ω)θ(Ω^Ω^Ω^Ω))),
[1 [1 ¬ 2 [1♦1♦2] 2] 2] has level θ(Ω^(Ω^Ω^Ω+1)),

[1 [1 ¬ 1 ¬ 2 [1♦1♦2] 2] 2] has level θ(Ω^(Ω^Ω^Ω+Ω)),
[1 [1 ¬ 1 ¬ 1 ¬ 2 [1♦1♦2] 2] 2] has level θ(Ω^(Ω^Ω^Ω+Ω^2)),
[1 [1 [2♦2] 2 [1♦1♦2] 2] 2] has level θ(Ω^(Ω^Ω^Ω+Ω^ω)),
[1 [1 [1♦3] 2 [1♦1♦2] 2] 2] has level θ(Ω^(Ω^Ω^Ω+Ω^Ω)),
[1 [1 [1♦1,2] 2 [1♦1♦2] 2] 2] has level θ(Ω^(Ω^Ω^Ω+Ω^Ω^ω)),
[1 [1 [1♦1♦2] 3] 2] has level θ(Ω^((Ω^Ω^Ω)2)),
[1 [1 [1♦1♦2] 1 \ 2] 2] has level θ(Ω^((Ω^Ω^Ω)θ(1))),
[1 [1 [1♦1♦2] 1 [1 [1♦1♦2] 2] 2] 2] has level θ(Ω^((Ω^Ω^Ω)θ(Ω^Ω^Ω^Ω))),
[1 [1 [1♦1♦2] 1 [1 [1♦1♦2] 1 [1 [1♦1♦2] 2] 2] 2] 2] has level
θ(Ω^((Ω^Ω^Ω)θ(Ω^((Ω^Ω^Ω)θ(Ω^Ω^Ω^Ω))))).

[1 [1 [1♦1♦2] 1 ¬ 2] 2] has level θ(Ω^Ω^(Ω^Ω+1)),
[1 [1 [1♦1♦2] 1 ¬ 1 ¬ 2] 2] has level θ(Ω^Ω^(Ω^Ω+2)),
[1 [1 [1♦1♦2] 1 [2♦2] 2] 2] has level θ(Ω^Ω^(Ω^Ω+ω)),
[1 [1 [1♦1♦2] 1 [1♦3] 2] 2] has level θ(Ω^Ω^(Ω^Ω+Ω)),
[1 [1 [1♦1♦2] 1 [1♦1,2] 2] 2] has level θ(Ω^Ω^(Ω^Ω+Ω^ω)),
[1 [1 [1♦1♦2] 1 [1♦1♦2] 2] 2] has level θ(Ω^Ω^((Ω^Ω)2)),
[1 [1 [2♦1♦2] 2] 2] has level θ(Ω^Ω^((Ω^Ω)ω)),
[1 [1 [1 \ 2 ♦1♦2] 2] 2] has level θ(Ω^Ω^((Ω^Ω)θ(1))),
[1 [1 [1 [1 [1♦1♦2] 2] 2 ♦1♦2] 2] 2] has level θ(Ω^Ω^((Ω^Ω)θ(Ω^Ω^Ω^Ω))),
[1 [1 [1 [1 [1 [1 [1♦1♦2] 2] 2 ♦1♦2] 2] 2 ♦1♦2] 2] 2] has level
θ(Ω^Ω^((Ω^Ω)θ(Ω^Ω^((Ω^Ω)θ(Ω^Ω^Ω^Ω))))).

[1 [1 [1♦2♦2] 2] 2] has level θ(Ω^Ω^Ω^(Ω+1)),
[1 [1 [1♦ 1 \ 2 ♦2] 2] 2] has level θ(Ω^Ω^Ω^(Ω+θ(1))),
[1 [1 [1♦ 1 [1 [1♦1♦2] 2] 2 ♦2] 2] 2] has level θ(Ω^Ω^Ω^(Ω+θ(Ω^Ω^Ω^Ω))),
[1 [1 [1♦ 1 [1 [1♦ 1 [1 [1♦1♦2] 2] 2 ♦2] 2] 2 ♦2] 2] 2] has level
θ(Ω^Ω^Ω^(Ω+θ(Ω^Ω^Ω^(Ω+θ(Ω^Ω^Ω^Ω))))),
[1 [1 [1♦1♦3] 2] 2] has level θ(Ω^Ω^Ω^(Ω2)),
[1 [1 [1♦1♦ 1 \ 2] 2] 2] has level θ(Ω^Ω^Ω^(Ωθ(1))),
[1 [1 [1♦1♦ 1 [1 [1♦1♦2] 2] 2] 2] 2] has level θ(Ω^Ω^Ω^(Ωθ(Ω^Ω^Ω^Ω))),
[1 [1 [1♦1♦ 1 [1 [1♦1♦ 1 [1 [1♦1♦2] 2] 2] 2] 2] 2] 2] has level
θ(Ω^Ω^Ω^(Ωθ(Ω^Ω^Ω^(Ωθ(Ω^Ω^Ω^Ω))))),
[1 [1 [1♦1♦1♦2] 2] 2] has level θ(Ω^Ω^Ω^Ω^2),
[1 [1 [1♦1♦1♦3] 2] 2] has level θ(Ω^Ω^Ω^((Ω^2)2)),
[1 [1 [1♦1♦1♦1♦2] 2] 2] has level θ(Ω^Ω^Ω^Ω^3),
[1 [1 [1♦1♦1♦1♦1♦2] 2] 2] has level θ(Ω^Ω^Ω^Ω^4),
[1 [1 [1♦1♦1♦...♦1♦2] 2] 2] (with n ♦ symbols) has level θ(Ω^Ω^Ω^Ω^(n-1)).

We next need a new symbol that follows the sequence: \ (backslash), ¬ (negation sign), ♦ (black diamond). It is taken to be ☼ (sun symbol) – the first 4-hyperseparator – and the second 3-hyperseparator in the series (after ♦) would be [2☼2]. [1☼2] 'drops down' to ♦, just as [1♦2] 'drops down' to ¬ and [1¬2] 'drops down' to \. The ☼ symbol requires a minimum of four pairs of square brackets in order to allow usage in the 'base layer' of a main array (three pairs would make it a 1-hyperseparator, two a 2-hyperseparator and one a 3-hyperseparator).

Thus the array,

{a, b [1 [1 [1 [2☼2] 2] 2] 2] 2} = {a ‹0 [1 [1 [2☼2] 2] 2] 2› b}
= {a ‹b ‹b ‹b ‹1☼2› b› b› b› b}

= {a ‹b ‹b ‹ b ♦ b ♦ b ♦ ... ♦ b › b› b› b}
(with b b's within innermost angle brackets).

This is a θ(Ω^Ω^Ω^Ω^ω)-recursive function.

In general,

{a, b [1 [1 [1 [n+1 ☼2] 2] 2] 2] 2} = {a ‹0 [1 [1 [n+1 ☼2] 2] 2] 2› b}
= {a ‹b ‹b ‹b ‹n☼2› b› b› b› b},

where 'b ‹n☼2› b' = 'b ‹n-1☼2› b [n☼2] b ‹n-1☼2› b [n☼2] ... [n☼2] b ‹n-1☼2› b'
(with b 'b ‹n-1☼2› b' strings),
'b ‹1☼2› b' = 'b ♦ b ♦ b ♦ ... ♦ b' (with b b's),
'b ‹0☼2› b' = 'b'.

[1 [1 [1 [2☼2] 2] 2] 2] has level θ(Ω^Ω^Ω^Ω^ω),
[1 [1 [1 [2☼2] 2] 2] 1 \ 2] has level θ(Ω^Ω^Ω^Ω^ω+1),
[1 [1 [1 [2☼2] 2] 2] 1 [1 [1 [2☼2] 2] 2] 2] has level θ((Ω^Ω^Ω^Ω^ω)2),
[1 [2 [1 [2☼2] 2] 2] 2] has level θ((Ω^Ω^Ω^Ω^ω)ω),
[1 [1 \ 2 [1 [2☼2] 2] 2] 2] has level θ((Ω^Ω^Ω^Ω^ω)θ(1)),
[1 [1 [1 [1 [2☼2] 2] 2] 2 [1 [2☼2] 2] 2] 2] has level θ((Ω^Ω^Ω^Ω^ω)θ(Ω^Ω^Ω^Ω^ω)),
[1 [1 ¬ 2 [1 [2☼2] 2] 2] 2] has level θ(Ω^(Ω^Ω^Ω^ω+1)),
[1 [1 ¬ 1 ¬ 2 [1 [2☼2] 2] 2] 2] has level θ(Ω^(Ω^Ω^Ω^ω+Ω)),
[1 [1 [2♦2] 2 [1 [2☼2] 2] 2] 2] has level θ(Ω^(Ω^Ω^Ω^ω+Ω^ω)),
[1 [1 [1♦3] 2 [1 [2☼2] 2] 2] 2] has level θ(Ω^(Ω^Ω^Ω^ω+Ω^Ω)),
[1 [1 [1♦1♦2] 2 [1 [2☼2] 2] 2] 2] has level θ(Ω^(Ω^Ω^Ω^ω+Ω^Ω^Ω)),
[1 [1 [1 [2☼2] 2] 3] 2] has level θ(Ω^((Ω^Ω^Ω^ω)2)),
[1 [1 [1 [2☼2] 2] 1 \ 2] 2] has level θ(Ω^((Ω^Ω^Ω^ω)θ(1))),
[1 [1 [1 [2☼2] 2] 1 [1 [1 [2☼2] 2] 2] 2] 2] has level θ(Ω^((Ω^Ω^Ω^ω)θ(Ω^Ω^Ω^Ω^ω))).

[1 [1 [1 [2☼2] 2] 1 ¬ 2] 2] has level θ(Ω^Ω^(Ω^Ω^ω+1)),
[1 [1 [1 [2☼2] 2] 1 ¬ 1 ¬ 2] 2] has level θ(Ω^Ω^(Ω^Ω^ω+2)),
[1 [1 [1 [2☼2] 2] 1 [2♦2] 2] 2] has level θ(Ω^Ω^(Ω^Ω^ω+ω)),
[1 [1 [1 [2☼2] 2] 1 [1♦3] 2] 2] has level θ(Ω^Ω^(Ω^Ω^ω+Ω)),
[1 [1 [1 [2☼2] 2] 1 [1♦1♦2] 2] 2] has level θ(Ω^Ω^(Ω^Ω^ω+Ω^Ω)),
[1 [1 [1 [2☼2] 2] 1 [1 [2☼2] 2] 2] 2] has level θ(Ω^Ω^((Ω^Ω^ω)2)),
[1 [1 [2 [2☼2] 2] 2] 2] has level θ(Ω^Ω^((Ω^Ω^ω)ω)),
[1 [1 [1 \ 2 [2☼2] 2] 2] 2] has level θ(Ω^Ω^((Ω^Ω^ω)θ(1))),
[1 [1 [1 [1 [1 [2☼2] 2] 2] 2 [2☼2] 2] 2] 2] has level θ(Ω^Ω^((Ω^Ω^ω)θ(Ω^Ω^Ω^Ω^ω))),
[1 [1 [1♦2 [2☼2] 2] 2] 2] has level θ(Ω^Ω^Ω^(Ω^ω+1)),
[1 [1 [1♦1♦2 [2☼2] 2] 2] 2] has level θ(Ω^Ω^Ω^(Ω^ω+Ω)),
[1 [1 [1 [2☼2] 3] 2] 2] has level θ(Ω^Ω^Ω^((Ω^ω)2)),
[1 [1 [1 [2☼2] 1 \ 2] 2] 2] has level θ(Ω^Ω^Ω^((Ω^ω)θ(1))),
[1 [1 [1 [2☼2] 1 [1 [1 [2☼2] 2] 2] 2] 2] 2] has level θ(Ω^Ω^Ω^((Ω^ω)θ(Ω^Ω^Ω^Ω^ω))).

If [X] and [Y] are 3-hyperseparators (♦ or arrays separated by ☼ symbols) such that [1 [1 [1 [X] 2] 2] 2] and [1 [1 [1 [Y] 2] 2] 2] have levels of θ(Ω^Ω^Ω^Ω^α) and θ(Ω^Ω^Ω^Ω^β) respectively (where α ≥ β), then [1 [1 [1 [X] 1 [Y] 2] 2] 2] would have level θ(Ω^Ω^Ω^Ω^(α+β)).

[1 [1 [1 [2☼2] 1♦2] 2] 2] has level θ(Ω^Ω^Ω^Ω^(ω+1)),
[1 [1 [1 [2☼2] 1♦1♦2] 2] 2] has level θ(Ω^Ω^Ω^Ω^(ω+2)),
[1 [1 [1 [2☼2] 1 [2☼2] 2] 2] 2] has level θ(Ω^Ω^Ω^Ω^(ω2)),

[1 [1 [1 [3☼2] 2] 2] 2] has level θ(Ω^Ω^Ω^Ω^ω^2),
[1 [1 [1 [1,2 ☼2] 2] 2] 2] has level θ(Ω^Ω^Ω^Ω^ω^ω),
[1 [1 [1 [1 \ 2 ☼2] 2] 2] 2] has level θ(Ω^Ω^Ω^Ω^θ(1)),
[1 [1 [1 [1 [1¬3] 2 ☼2] 2] 2] 2] has level θ(Ω^Ω^Ω^Ω^θ(Ω)),
[1 [1 [1 [1 [1¬1¬2] 2 ☼2] 2] 2] 2] has level θ(Ω^Ω^Ω^Ω^θ(Ω^Ω)),
[1 [1 [1 [1 [1 [1♦3] 2] 2 ☼2] 2] 2] 2] has level θ(Ω^Ω^Ω^Ω^θ(Ω^Ω^Ω)),
[1 [1 [1 [1 [1 [1 [2☼2] 2] 2] 2 ☼2] 2] 2] 2] has level θ(Ω^Ω^Ω^Ω^θ(Ω^Ω^Ω^Ω)),
[1 [1 [1 [1 [1 [1 [1 \ 2☼2] 2] 2] 2 ☼2] 2] 2] 2] has level θ(Ω^Ω^Ω^Ω^θ(Ω^Ω^Ω^Ω^θ(1))),
[1 [1 [1 [1 [1 [1 [1 [1 [1 [1 \ 2 ☼2] 2] 2] 2 ☼2] 2] 2] 2 ☼2] 2] 2] 2] has level
θ(Ω^Ω^Ω^Ω^θ(Ω^Ω^Ω^Ω^θ(Ω^Ω^Ω^Ω^θ(1)))).

[1 [1 [1 [1☼3] 2] 2] 2] has level θ(Ω^Ω^Ω^Ω^Ω),
[1 [1 [1 [1☼3] 1♦2] 2] 2] has level θ(Ω^Ω^Ω^Ω^(Ω+1)),
[1 [1 [1 [1☼3] 1 [2☼2] 2] 2] 2] has level θ(Ω^Ω^Ω^Ω^(Ω+ω)),
[1 [1 [1 [1☼3] 1 [1 \ 2 ☼2] 2] 2] 2] has level θ(Ω^Ω^Ω^Ω^(Ω+θ(1))),
[1 [1 [1 [1☼3] 1 [1 [1 [1 [1☼3] 2] 2] 2 ☼2] 2] 2] 2] has level
θ(Ω^Ω^Ω^Ω^(Ω+θ(Ω^Ω^Ω^Ω^Ω))),
[1 [1 [1 [1☼3] 1 [1☼3] 2] 2] 2] has level θ(Ω^Ω^Ω^Ω^(Ω2)),
[1 [1 [1 [2☼3] 2] 2] 2] has level θ(Ω^Ω^Ω^Ω^(Ωω)),
[1 [1 [1 [1 \ 2 ☼3] 2] 2] 2] has level θ(Ω^Ω^Ω^Ω^(Ωθ(1))),
[1 [1 [1 [1 [1 [1 [1☼3] 2] 2] 2 ☼3] 2] 2] 2] has level θ(Ω^Ω^Ω^Ω^(Ωθ(Ω^Ω^Ω^Ω^Ω))),
[1 [1 [1 [1 [1 [1 [1 [1 [1 [1☼3] 2] 2] 2 ☼3] 2] 2] 2 ☼3] 2] 2] 2] has level
θ(Ω^Ω^Ω^Ω^(Ωθ(Ω^Ω^Ω^Ω^(Ωθ(Ω^Ω^Ω^Ω^Ω))))).

[1 [1 [1 [1☼4] 2] 2] 2] has level θ(Ω^Ω^Ω^Ω^Ω^2),
[1 [1 [1 [1☼5] 2] 2] 2] has level θ(Ω^Ω^Ω^Ω^Ω^3),
[1 [1 [1 [1☼1,2] 2] 2] 2] has level θ(Ω^Ω^Ω^Ω^Ω^ω),
[1 [1 [1 [1 ☼ 1 \ 2] 2] 2] 2] has level θ(Ω^Ω^Ω^Ω^Ω^θ(1)),
[1 [1 [1 [1 ☼ 1 [1 [1 [1☼3] 2] 2] 2] 2] 2] 2] has level θ(Ω^Ω^Ω^Ω^Ω^θ(Ω^Ω^Ω^Ω^Ω)),
[1 [1 [1 [1 ☼ 1 [1 [1 [1 ☼ 1 \ 2] 2] 2] 2] 2] 2] 2] has level
θ(Ω^Ω^Ω^Ω^Ω^θ(Ω^Ω^Ω^Ω^Ω^θ(1))),
[1 [1 [1 [1 ☼ 1 [1 [1 [1 ☼ 1 [1 [1 [1 ☼ 1 \ 2] 2] 2] 2] 2] 2] 2] 2] 2] 2] has level
θ(Ω^Ω^Ω^Ω^Ω^θ(Ω^Ω^Ω^Ω^Ω^θ(Ω^Ω^Ω^Ω^Ω^θ(1)))),
[1 [1 [1 [1 ☼ 1 ☼ 2] 2] 2] 2] has level θ(Ω^Ω^Ω^Ω^Ω^Ω).

Notice some similarities of the 8 separators above with the 8 below:

[1 [1 [1♦4] 2] 2] has level θ(Ω^Ω^Ω^2),
[1 [1 [1♦5] 2] 2] has level θ(Ω^Ω^Ω^3),
[1 [1 [1♦1,2] 2] 2] has level θ(Ω^Ω^Ω^ω),
[1 [1 [1 ♦ 1 \ 2] 2] 2] has level θ(Ω^Ω^Ω^θ(1)),
[1 [1 [1 ♦ 1 [1 [1♦3] 2] 2] 2] 2] has level θ(Ω^Ω^Ω^θ(Ω^Ω^Ω)),
[1 [1 [1 ♦ 1 [1 [1 ♦ 1 \ 2] 2] 2] 2] 2] has level θ(Ω^Ω^Ω^θ(Ω^Ω^Ω^θ(1))),
[1 [1 [1 ♦ 1 [1 [1 ♦ 1 [1 [1 ♦ 1 \ 2] 2] 2] 2] 2] 2] 2] has level
θ(Ω^Ω^Ω^θ(Ω^Ω^Ω^θ(Ω^Ω^Ω^θ(1)))),
[1 [1 [1 ♦ 1 ♦ 2] 2] 2] has level θ(Ω^Ω^Ω^Ω).

The creation of the first group of 8 separators mirrors the second group of 8, except that the ♦'s are ☼'s, the double nested square brackets become treble nested square brackets and the Ω^Ω^Ω's in the θ function become Ω^Ω^Ω^Ω^Ω's. Another pattern we are spotting is that when X and Y are arrays such that [1 [1 [X♦Y] 2] 2] has level θ(Ω^Ω^α), then [1 [1 [1 [X☼Y] 2] 2] 2] would have level θ(Ω^Ω^Ω^Ω^α).

A 3-hyperseparator (e.g. ♦ symbol or [X☼Y]) cannot be on the same ‘nested level’ as a 2-hyperseparator (e.g. ¬ symbol, [X♦Y] or [W [X☼Y] Z]). A 4-hyperseparator (the lowest being the ☼ symbol) cannot be on the same ‘nested level’ as a 2- or 3-hyperseparator. But there are no restrictions on the appearance of normal separators (0-hyperseparators) or 1-hyperseparators. Suppose that k separators [X_1], [X_2], ... , [X_k], where each [X_i] is a p_i-hyperseparator, appear on the same ‘nested level’ somewhere within a giant normal separator [N], as in this example

[N] = [# [n_1 [X_1] n_2 [X_2] ... [X_k] n_{k+1}] #*]   (# and #* represent the remainder of N).

All of the p_i can be either 0 or 1, but if any of the p_i is greater than one, say $p_r = x > 1$ for some $1 \le r \le k$, then each of the p_i can take one of only three values – 0, 1 and x – which would mean that [n_1 [X_1] n_2 [X_2] ... [X_k] n_{k+1}] would be an (x-1)-hyperseparator.

The θ(Ω^Ω^Ω^Ω^Ω) level separator

{a, b [1 [1 [1 [1☼3] 2] 2] 2] 2} = {a ‹0 [1 [1 [1☼3] 2] 2] 2› b}
= {a ‹b ‹R_b› b› b},

where R_n = ‘b ‹b ‹b ‹R_{n-1}› b ☼2› b› b’,
R_1 = ‘0’.

For example,

{3, 2 [1 [1 [1 [1☼3] 2] 2] 2] 2} = {3 ‹0 [1 [1 [1☼3] 2] 2] 2› 2}
= {3 ‹2 ‹2 ‹2 ‹2☼2› 2› 2› 2› 2}
{3, 3 [1 [1 [1 [1☼3] 2] 2] 2] 2} = {3 ‹0 [1 [1 [1☼3] 2] 2] 2› 3}
= {3 ‹3 ‹3 ‹3 ‹3 ‹3 ‹3 ‹3☼2› 3› 3› 3☼2› 3› 3› 3› 3},
{3, 4 [1 [1 [1 [1☼3] 2] 2] 2] 2} = {3 ‹0 [1 [1 [1☼3] 2] 2] 2› 4}
= {3 ‹4 ‹4 ‹4 ‹4 ‹4 ‹4 ‹4 ‹4 ‹4 ‹4☼2› 4› 4› 4☼2› 4› 4› 4☼2› 4› 4› 4› 4}.

The [1 [1 [1 [1☼3] 2] 2] 2] separator marks the beginning of level 3 of my Nested Hyper-Nested Array Notation, just as [1 [1¬3] 2] and [1 [1 [1♦3] 2] 2] mark the beginning of levels 1 and 2 respectively. The nesting separator [1 [1 [1☼3] 2] 2] is the lowest special hyperseparator to require three pairs of brackets per iteration of the R_n function, and thus can be regarded as a 3-nesting separator.

The θ(Ω^Ω^Ω^Ω^Ω^(k-2)) level separator

{a, b [1 [1 [1 [1☼k] 2] 2] 2] 2} = {a ‹0 [1 [1 [1☼k] 2] 2] 2› b}
= {a ‹b ‹R_b› b› b},

where R_n = ‘b ‹b ‹b ‹R_{n-1}› b ☼k-1› b› b’,
R_1 = ‘0’.

The θ(Ω^Ω^Ω^Ω^Ω^ω) level separator

{a, b [1 [1 [1 [1☼1,2] 2] 2] 2] 2} = {a ‹0 [1 [1 [1☼1,2] 2] 2] 2› b}
= {a ‹b ‹b ‹b ‹b☼b› b› b› b› b}.

We now consider the various scenarios for the [A_{1,p_1}] separator in the initial equation of Rule A5, which reads:

‘a ‹ 0 [$A_{1,1}$] 1 [$A_{1,2}$] ... 1 [A_{1,p_1}] c_1 $\#_1$ $\#_0$ › b’ = ‘a ‹S_1 $\#_0$› b’,

where each of the [$A_{1,j}$] are either normal separators or 1-hyperseparators.

Nesting separators (Rules A5b or A5c) include the backslash (\) and hyperseparators of the form

[1 [B_1] 1 [B_2] ... 1 [B_q] d #*],

where each of the [B_j] are 2-hyperseparators, with [B_q] being a nesting 2-hyperseparator, which includes the ¬ sign and 2-hyperseparators of the form

[1 [C_1] 1 [C_2] ... 1 [C_r] e #**],

where each of the [C_j] are 3-hyperseparators (either ♦ symbol or of the form [X☼Y], where X and Y are arrays), with [C_r] being a nesting 3-hyperseparator, which includes the ♦ symbol and 3-hyperseparators of the form

[1 ☼ k #***],

where $q \geq 1$, $r \geq 1$, $d \geq 2$, $e \geq 2$, $k \geq 2$ and #*, #** and #*** are rest of their respective arrays.

If [A_{1,p_1}] is a backslash (\), Rule A5b is executed. If [A_{1,p_1}] is any other nesting separator, A5c is processed. Rule A5c now has three subrules. (The i-subscripts have been removed.)

Rule A5c1 ([A_p] = [1 [B_1] 1 [B_2] ... 1 [B_{q-1}] 1 ¬ d #*],
where $q \geq 1$, $d \geq 2$):

S = ‘b ‹A_1’› b [A_1] b ‹A_2’› b [A_2] ... b ‹A_{p-1}’› b [A_{p-1}] b ‹R_b› b [A_p] c-1 #’,
R_n = ‘b ‹B_1’› b [B_1] b ‹B_2’› b [B_2] ... b ‹B_{q-1}’› b [B_{q-1}]
b ‹A_1’› b [A_1] b ‹A_2’› b [A_2] ... b ‹A_{p-1}’› b [A_{p-1}] b ‹R_{n-1}› b [A_p] c-1 # ¬ d-1 #*’ ($n > 1$),
R_1 = ‘0’.

Rule A5c2 ([A_p] = [1 [B_1] 1 [B_2] ... 1 [B_q] d #*],
where [B_q] = [1 [C_1] 1 [C_2] ... 1 [C_{r-1}] 1 ♦ e #**]
and $q \geq 1$, $r \geq 1$, $d \geq 2$, $e \geq 2$):

S = ‘b ‹A_1’› b [A_1] b ‹A_2’› b [A_2] ... b ‹A_{p-1}’› b [A_{p-1}] b ‹R_b› b [A_p] c-1 #’,
R_n = ‘b ‹B_1’› b [B_1] b ‹B_2’› b [B_2] ... b ‹B_{q-1}’› b [B_{q-1}] b ‹T_n› b [B_q] d-1 #*’ ($n > 1$),
T_n = ‘b ‹C_1’› b [C_1] b ‹C_2’› b [C_2] ... b ‹C_{r-1}’› b [C_{r-1}]
b ‹A_1’› b [A_1] b ‹A_2’› b [A_2] ... b ‹A_{p-1}’› b [A_{p-1}] b ‹R_{n-1}› b [A_p] c-1 # ♦ e-1 #**’ ($n > 1$),
R_1 = ‘0’.

Rule A5c3 ([A_p] = [1 [B_1] 1 [B_2] ... 1 [B_q] d #*],
where [B_q] = [1 [C_1] 1 [C_2] ... 1 [C_r] e #**],
where [C_r] = [1 ☼ k #***]
and $q \geq 1$, $r \geq 1$, $d \geq 2$, $e \geq 2$, $k \geq 2$):

S = ‘b ‹A_1’› b [A_1] b ‹A_2’› b [A_2] ... b ‹A_{p-1}’› b [A_{p-1}] b ‹R_b› b [A_p] c-1 #’,
R_n = ‘b ‹B_1’› b [B_1] b ‹B_2’› b [B_2] ... b ‹B_{q-1}’› b [B_{q-1}] b ‹T_n› b [B_q] d-1 #*’ ($n > 1$),
T_n = ‘b ‹C_1’› b [C_1] b ‹C_2’› b [C_2] ... b ‹C_{r-1}’› b [C_{r-1}] b ‹U_n› b [C_r] e-1 #**’ ($n > 1$),
U_n = ‘b ‹A_1’› b [A_1] b ‹A_2’› b [A_2] ... b ‹A_{p-1}’› b [A_{p-1}] b ‹R_{n-1}› b [A_p] c-1 # ☼ k-1 #***’ ($n > 1$),
R_1 = ‘0’.

In a nutshell, subrule n of Rule A5c processes n-nesting separators.

Plugging separators (Rule A5d) include hyperseparators of the form

[1 [B_1] 1 [B_2] ... 1 [B_q] d #*],

where [B_q] = [X [C_1] Y], X ≠ ‘1’, Y ≠ ‘1’,
in one set, and those of the form

[1 [B_1] 1 [B_2] ... 1 [B_q] d #*],

where [B_q] = [1 [C_1] 1 [C_2] ... 1 [C_r] e #**],
where [C_r] = [X☼Y], X ≠ ‘1’, Y ≠ ‘1’,

in another set. In both sets, each of the [B_j] are 2-hyperseparators, each of the [C_j] are 3-hyperseparators (where [C_1] is the first 3-hyperseparator in the ‘base layer’ of [B_q]), X and Y are strings (X does not contain any 2- or higher order hyperseparators in its ‘base layer’, whereas Y may contain some), q ≥ 1, r ≥ 1, d ≥ 2, e ≥ 2 and #* and #** are rest of their respective arrays.

Rule A5d splits into two subrules – A5d1 for the first set (2-plugging separators) and A5d2 for the latter (3-plugging separators). (The i-subscripts have been removed as in Rules A5c1-3.)

Rule A5d1 ([A_p] = [1 [B_1] 1 [B_2] ... 1 [B_q] d #*],
where [B_q] = [X [C_1] Y], X ≠ ‘1’, Y ≠ ‘1’
and q ≥ 1, d ≥ 2):
S = ‘b ‹A_1’› b [A_1] b ‹A_2’› b [A_2] ... b ‹A_{p-1}’› b [A_{p-1}] b ‹T› b [A_p] c-1 #’,
T = ‘b ‹B_1’› b [B_1] b ‹B_2’› b [B_2] ... b ‹B_q’› b [B_q] d-1 #*’.

Rule A5d2 ([A_p] = [1 [B_1] 1 [B_2] ... 1 [B_q] d #*],
where [B_q] = [1 [C_1] 1 [C_2] ... 1 [C_r] e #**],
where [C_r] = [X☼Y], X ≠ ‘1’, Y ≠ ‘1’
and q ≥ 1, r ≥ 1, d ≥ 2, e ≥ 2):
S = ‘b ‹A_1’› b [A_1] b ‹A_2’› b [A_2] ... b ‹A_{p-1}’› b [A_{p-1}] b ‹T› b [A_p] c-1 #’,
T = ‘b ‹B_1’› b [B_1] b ‹B_2’› b [B_2] ... b ‹B_{q-1}’› b [B_{q-1}] b ‹U› b [B_q] d-1 #*’,
U = ‘b ‹C_1’› b [C_1] b ‹C_2’› b [C_2] ... b ‹C_r’› b [C_r] e-1 #**’.

Branching separators (Rule A5a) now come in three sets. The first set contains hyperseparators of the form
[1 [B_1] 1 [B_2] ... 1 [B_q] 1 [A_1] 1 [A_2] ... 1 [A_p] k #*].
The second set comprises hyperseparators of the form
[1 [B_1] 1 [B_2] ... 1 [B_q] d #*],
where [B_q] = [1 [C_1] 1 [C_2] ... 1 [C_r] 1 [A_1] 1 [A_2] ... 1 [A_p] k #**].
The third set comprises hyperseparators of the form
[1 [B_1] 1 [B_2] ... 1 [B_q] d #*],
where [B_q] = [1 [C_1] 1 [C_2] ... 1 [C_r] e #**],
where [C_r] = [1 ☼ 1 [A_1] 1 [A_2] ... 1 [A_p] k #***].
In all three sets, each of the [A_j] are either normal separators or 1-hyperseparators, each of the [B_j] are 2-hyperseparators, each of the [C_j] are 3-hyperseparators, p ≥ 1, q ≥ 1, r ≥ 1, d ≥ 2, e ≥ 2, k ≥ 2 and #*, #** and #*** are rest of their respective arrays.

Rule A5a now has three subrules, with subrule n for n-branching separators (nth set of the three shown above). Note that i is initially set to 1.

Rule A5a1 ([A_{i,p_i}] = [1 [$B_{i,1}$] 1 [$B_{i,2}$] ... 1 [B_{i,q_i}] 1 [$A_{i+1,1}$] 1 [$A_{i+1,2}$] ... 1 [$A_{i+1,p_{i+1}}$] c_{i+1} $\#_{i+1}$],
where p_{i+1} ≥ 1, q_i ≥ 1):
S_i = ‘b ‹$A_{i,1}$’› b [$A_{i,1}$] b ‹$A_{i,2}$’› b [$A_{i,2}$] ... b ‹A_{i,p_i-1}’› b [A_{i,p_i-1}] b ‹T_i› b [A_{i,p_i}] c_i-1 $\#_i$’,
T_i = ‘b ‹$B_{i,1}$’› b [$B_{i,1}$] b ‹$B_{i,2}$’› b [$B_{i,2}$] ... b ‹B_{i,q_i}’› b [B_{i,q_i}] S_{i+1}’.
Increment i by 1 and repeat Rules A5a-e.

Rule A5a2 ([A_{i,p_i}] = [1 [$B_{i,1}$] 1 [$B_{i,2}$] ... 1 [B_{i,q_i}] d_i #*$_i$],
where [B_{i,q_i}] = [1 [$C_{i,1}$] 1 [$C_{i,2}$] ... 1 [C_{i,r_i}] 1 [$A_{i+1,1}$] 1 [$A_{i+1,2}$] ... 1 [$A_{i+1,p_{i+1}}$] c_{i+1} #$_{i+1}$]
and $p_{i+1} \geq 1$, $q_i \geq 1$, $r_i \geq 1$):

S_i = ‘b ‹$A_{i,1}$’› b [$A_{i,1}$] b ‹$A_{i,2}$’› b [$A_{i,2}$] ... b ‹A_{i,p_i-1}’› b [A_{i,p_i-1}] b ‹T_i› b [A_{i,p_i}] c_i-1 #$_i$’,
T_i = ‘b ‹$B_{i,1}$’› b [$B_{i,1}$] b ‹$B_{i,2}$’› b [$B_{i,2}$] ... b ‹B_{i,q_i-1}’› b [B_{i,q_i-1}] b ‹U_i› b [B_{i,q_i}] d_i-1 #*$_i$’,
U_i = ‘b ‹$C_{i,1}$’› b [$C_{i,1}$] b ‹$C_{i,2}$’› b [$C_{i,2}$] ... b ‹C_{i,r_i}’› b [C_{i,r_i}] S_{i+1}’.

Increment i by 1 and repeat Rules A5a-e.

Rule A5a3 ([A_{i,p_i}] = [1 [$B_{i,1}$] 1 [$B_{i,2}$] ... 1 [B_{i,q_i}] d_i #*$_i$],
where [B_{i,q_i}] = [1 [$C_{i,1}$] 1 [$C_{i,2}$] ... 1 [C_{i,r_i}] e_i #**$_i$],
where [C_{i,r_i}] = [1 ☼ 1 [$A_{i+1,1}$] 1 [$A_{i+1,2}$] ... 1 [$A_{i+1,p_{i+1}}$] c_{i+1} #$_{i+1}$]
and $p_{i+1} \geq 1$, $q_i \geq 1$, $r_i \geq 1$):

S_i = ‘b ‹$A_{i,1}$’› b [$A_{i,1}$] b ‹$A_{i,2}$’› b [$A_{i,2}$] ... b ‹A_{i,p_i-1}’› b [A_{i,p_i-1}] b ‹T_i› b [A_{i,p_i}] c_i-1 #$_i$’,
T_i = ‘b ‹$B_{i,1}$’› b [$B_{i,1}$] b ‹$B_{i,2}$’› b [$B_{i,2}$] ... b ‹B_{i,q_i-1}’› b [B_{i,q_i-1}] b ‹U_i› b [B_{i,q_i}] d_i-1 #*$_i$’,
U_i = ‘b ‹$C_{i,1}$’› b [$C_{i,1}$] b ‹$C_{i,2}$’› b [$C_{i,2}$] ... b ‹b ☼ S_{i+1}› b [C_{i,r_i}] e_i-1 #**$_i$’.

Increment i by 1 and repeat Rules A5a-e.

We now extend the separators to multiple sun symbols on the same ‘nested level’.

[1 [1 [1 [1☼1☼2] 2] 2] 2] has level θ(Ω^Ω^Ω^Ω^Ω^Ω),
[1 [1 [1 [1☼1☼2] 2] 2] 1 [1 [1 [1☼1☼2] 2] 2] 2] has level θ((Ω^Ω^Ω^Ω^Ω^Ω)2),
[1 [2 [1 [1☼1☼2] 2] 2] 2] has level θ((Ω^Ω^Ω^Ω^Ω^Ω)ω),
[1 [1 \ 2 [1 [1☼1☼2] 2] 2] 2] has level θ((Ω^Ω^Ω^Ω^Ω^Ω)θ(1)),
[1 [1 [1 [1 [1☼1☼2] 2] 2] 2 [1 [1☼1☼2] 2] 2] 2] has level
θ((Ω^Ω^Ω^Ω^Ω^Ω)θ(Ω^Ω^Ω^Ω^Ω^Ω)),
[1 [1 ¬ 2 [1 [1☼1☼2] 2] 2] 2] has level θ(Ω^(Ω^Ω^Ω^Ω^Ω + 1)),
[1 [1 ¬ 1 ¬ 2 [1 [1☼1☼2] 2] 2] 2] has level θ(Ω^(Ω^Ω^Ω^Ω^Ω + Ω)),
[1 [1 [1♦3] 2 [1 [1☼1☼2] 2] 2] 2] has level θ(Ω^(Ω^Ω^Ω^Ω^Ω + Ω^Ω)),
[1 [1 [1♦1♦2] 2 [1 [1☼1☼2] 2] 2] 2] has level θ(Ω^(Ω^Ω^Ω^Ω^Ω + Ω^Ω^Ω)),
[1 [1 [1 [1☼3] 2] 2 [1 [1☼1☼2] 2] 2] 2] has level θ(Ω^(Ω^Ω^Ω^Ω^Ω + Ω^Ω^Ω^Ω)),
[1 [1 [1 [1☼1☼2] 2] 3] 2] has level θ(Ω^((Ω^Ω^Ω^Ω^Ω)2)),
[1 [1 [1 [1☼1☼2] 2] 1 \ 2] 2] has level θ(Ω^((Ω^Ω^Ω^Ω^Ω)θ(1))),
[1 [1 [1 [1☼1☼2] 2] 1 [1 [1 [1☼1☼2] 2] 2] 2] 2] has level
θ(Ω^((Ω^Ω^Ω^Ω^Ω)θ(Ω^Ω^Ω^Ω^Ω^Ω))).

[1 [1 [1 [1☼1☼2] 2] 1 ¬ 2] 2] has level θ(Ω^Ω^(Ω^Ω^Ω^Ω + 1)),
[1 [1 [1 [1☼1☼2] 2] 1 [1♦3] 2] 2] has level θ(Ω^Ω^(Ω^Ω^Ω^Ω + Ω)),
[1 [1 [1 [1☼1☼2] 2] 1 [1♦1♦2] 2] 2] has level θ(Ω^Ω^(Ω^Ω^Ω^Ω + Ω^Ω)),
[1 [1 [1 [1☼1☼2] 2] 1 [1 [1☼3] 2] 2] 2] has level θ(Ω^Ω^(Ω^Ω^Ω^Ω + Ω^Ω^Ω)),
[1 [1 [1 [1☼1☼2] 2] 1 [1 [1☼1☼2] 2] 2] 2] has level θ(Ω^Ω^((Ω^Ω^Ω^Ω)2)),
[1 [1 [2 [1☼1☼2] 2] 2] 2] has level θ(Ω^Ω^((Ω^Ω^Ω^Ω)ω)),
[1 [1 [1 \ 2 [1☼1☼2] 2] 2] 2] has level θ(Ω^Ω^((Ω^Ω^Ω^Ω)θ(1))),
[1 [1 [1 [1 [1 [1☼1☼2] 2] 2] 2 [1☼1☼2] 2] 2] 2] has level
θ(Ω^Ω^((Ω^Ω^Ω^Ω)θ(Ω^Ω^Ω^Ω^Ω^Ω))),
[1 [1 [1 ♦ 2 [1☼1☼2] 2] 2] 2] has level θ(Ω^Ω^Ω^(Ω^Ω^Ω + 1)),
[1 [1 [1 ♦ 1 ♦ 2 [1☼1☼2] 2] 2] 2] has level θ(Ω^Ω^Ω^(Ω^Ω^Ω + Ω)),
[1 [1 [1 [1☼3] 2 [1☼1☼2] 2] 2] 2] has level θ(Ω^Ω^Ω^(Ω^Ω^Ω + Ω^Ω)),
[1 [1 [1 [1☼1☼2] 3] 2] 2] has level θ(Ω^Ω^Ω^((Ω^Ω^Ω)2)),
[1 [1 [1 [1☼1☼2] 1 \ 2] 2] 2] has level θ(Ω^Ω^Ω^((Ω^Ω^Ω)θ(1))),

[1 [1 [1 [1☼1☼2] 1 [1 [1 [1☼1☼2] 2] 2] 2] 2] 2] has level
θ(Ω^Ω^Ω^((Ω^Ω^Ω)θ(Ω^Ω^Ω^Ω^Ω^Ω))).

[1 [1 [1 [1☼1☼2] 1 ♦ 2] 2] 2] has level θ(Ω^Ω^Ω^Ω^(Ω^Ω + 1)),
[1 [1 [1 [1☼1☼2] 1 [1☼3] 2] 2] 2] has level θ(Ω^Ω^Ω^Ω^(Ω^Ω + Ω)),
[1 [1 [1 [1☼1☼2] 1 [1☼1☼2] 2] 2] 2] has level θ(Ω^Ω^Ω^Ω^((Ω^Ω)2)),
[1 [1 [1 [2☼1☼2] 2] 2] 2] has level θ(Ω^Ω^Ω^Ω^((Ω^Ω)ω)),
[1 [1 [1 [1 \ 2 ☼1☼2] 2] 2] 2] has level θ(Ω^Ω^Ω^Ω^((Ω^Ω)θ(1))),
[1 [1 [1 [1 [1 [1 [1☼1☼2] 2] 2] 2 ☼1☼2] 2] 2] 2] has level
θ(Ω^Ω^Ω^Ω^((Ω^Ω)θ(Ω^Ω^Ω^Ω^Ω^Ω))),
[1 [1 [1 [1☼2☼2] 2] 2] 2] has level θ(Ω^Ω^Ω^Ω^Ω^(Ω+1)),
[1 [1 [1 [1☼ 1 \ 2 ☼2] 2] 2] 2] has level θ(Ω^Ω^Ω^Ω^Ω^(Ω+θ(1))),
[1 [1 [1 [1☼ 1 [1 [1 [1☼1☼2] 2] 2] 2 ☼2] 2] 2] 2] has level
θ(Ω^Ω^Ω^Ω^Ω^(Ω+θ(Ω^Ω^Ω^Ω^Ω^Ω))),
[1 [1 [1 [1☼1☼3] 2] 2] 2] has level θ(Ω^Ω^Ω^Ω^Ω^(Ω2)),
[1 [1 [1 [1☼1☼ 1 \ 2] 2] 2] 2] has level θ(Ω^Ω^Ω^Ω^Ω^(Ωθ(1))),
[1 [1 [1 [1☼1☼ 1 [1 [1 [1☼1☼2] 2] 2] 2] 2] 2] 2] has level
θ(Ω^Ω^Ω^Ω^Ω^(Ωθ(Ω^Ω^Ω^Ω^Ω^Ω))),
[1 [1 [1 [1☼1☼1☼2] 2] 2] 2] has level θ(Ω^Ω^Ω^Ω^Ω^Ω^2),
[1 [1 [1 [1☼1☼1☼1☼2] 2] 2] 2] has level θ(Ω^Ω^Ω^Ω^Ω^Ω^3),
[1 [1 [1 [1☼1☼1☼...☼1☼2] 2] 2] 2] (with n ☼ symbols) has level θ(Ω^Ω^Ω^Ω^Ω^Ω^(n-1)).

At this stage we need to create a generalised n-hyperseparator symbol, which is represented in this paper by a backslash with a subscript ($\backslash_n$). The $\backslash_1$ symbol is the backslash with the 1 omitted, $\backslash_2$ represents the ¬ symbol, $\backslash_3$ is the ♦ symbol and $\backslash_4$ represents the ☼ symbol. The second 4-hyperseparator in the series (after $\backslash_4$ or ☼) would be [2 $\backslash_5$ 2]. In general, the first two n-hyperseparators are $\backslash_n$ and [2 $\backslash_{n+1}$ 2], since [1 $\backslash_n$ 2] 'drops down' to $\backslash_{n-1}$ (in fact, we can even write the ε_0 level separator [1 \ 2] as $\backslash_0$). The $\backslash_n$ symbol requires a minimum of n pairs of square brackets enclosing it in order to allow usage in the 'base layer' of a main array; $\backslash_n$ enclosed by m pairs of square brackets is an (n-m)-hyperseparator (when n > m) and a normal separator (or 0-hyperseparator) otherwise.

Using the backslash subscript system, the most salient separators introduced so far are:
[1] has level 0,
[2] has level 1,
[1, 2] has level ω,
[1 [2] 2] has level ω^ω,
[1 [1, 2] 2] has level ω^ω^ω,
[1 \ 2] has level θ(1) = ε_0,
[1 [1 \ 2] 2 \ 2] has level θ(1)2 = $\varepsilon_0$2,
[1 \ 3] has level θ(1, 1) = ε_1,
[1 \ 4] has level θ(1, 2) = ε_2,
[1 \ 1, 2] has level θ(1, ω) = ε_ω,
[1 \ 1 [1 \ 2] 2] has level θ(1, θ(1)) = ε(ε_0),
[1 \ 1 \ 2] has level θ(2) = ζ_0,
[1 \ 1 \ 1 \ 2] has level θ(3) = φ(3, 0),
[1 [2 $\backslash_2$ 2] 2] has level θ(ω) = φ(ω, 0),
[1 [1 \ 2 $\backslash_2$ 2] 2] has level θ(θ(1)) = φ(ε_0, 0),
[1 [1 $\backslash_2$ 3] 2] has level θ(Ω) = Γ_0,

[1 [1 $\backslash_2$ 4] 2] has level θ(Ω^2),
[1 [1 $\backslash_2$ 1, 2] 2] has level θ(Ω^ω) (small Veblen ordinal),
[1 [1 $\backslash_2$ 1 \ 2] 2] has level θ(Ω^θ(1)),
[1 [1 $\backslash_2$ 1 $\backslash_2$ 2] 2] has level θ(Ω^Ω) (large Veblen ordinal),
[1 [1 $\backslash_2$ 1 $\backslash_2$ 1 $\backslash_2$ 2] 2] has level θ(Ω^Ω^2),
[1 [1 [2 $\backslash_3$ 2] 2] 2] has level θ(Ω^Ω^ω),
[1 [1 [1 \ 2 $\backslash_3$ 2] 2] 2] has level θ(Ω^Ω^θ(1)),
[1 [1 [1 $\backslash_3$ 3] 2] 2] has level θ(Ω^Ω^Ω),
[1 [1 [1 $\backslash_3$ 4] 2] 2] has level θ(Ω^Ω^Ω^2),
[1 [1 [1 $\backslash_3$ 1, 2] 2] 2] has level θ(Ω^Ω^Ω^ω),
[1 [1 [1 $\backslash_3$ 1 \ 2] 2] 2] has level θ(Ω^Ω^Ω^θ(1)),
[1 [1 [1 $\backslash_3$ 1 $\backslash_3$ 2] 2] 2] has level θ(Ω^Ω^Ω^Ω),
[1 [1 [1 $\backslash_3$ 1 $\backslash_3$ 1 $\backslash_3$ 2] 2] 2] has level θ(Ω^Ω^Ω^Ω^2),
[1 [1 [1 [2 $\backslash_4$ 2] 2] 2] 2] has level θ(Ω^Ω^Ω^Ω^ω),
[1 [1 [1 [1 \ 2 $\backslash_4$ 2] 2] 2] 2] has level θ(Ω^Ω^Ω^Ω^θ(1)),
[1 [1 [1 [1 $\backslash_4$ 3] 2] 2] 2] has level θ(Ω^Ω^Ω^Ω^Ω),
[1 [1 [1 [1 $\backslash_4$ 4] 2] 2] 2] has level θ(Ω^Ω^Ω^Ω^Ω^2),
[1 [1 [1 [1 $\backslash_4$ 1, 2] 2] 2] 2] has level θ(Ω^Ω^Ω^Ω^Ω^ω),
[1 [1 [1 [1 $\backslash_4$ 1 \ 2] 2] 2] 2] has level θ(Ω^Ω^Ω^Ω^Ω^θ(1)),
[1 [1 [1 [1 $\backslash_4$ 1 $\backslash_4$ 2] 2] 2] 2] has level θ(Ω^Ω^Ω^Ω^Ω^Ω),
[1 [1 [1 [1 $\backslash_4$ 1 $\backslash_4$ 1 $\backslash_4$ 2] 2] 2] 2] has level θ(Ω^Ω^Ω^Ω^Ω^Ω^2).

Continuing beyond this, we would find that:
[1 [1 [1 [1 [2 $\backslash_5$ 2] 2] 2] 2] 2] has level θ(Ω^Ω^Ω^Ω^Ω^Ω^ω),
[1 [1 [1 [1 [1 \ 2 $\backslash_5$ 2] 2] 2] 2] 2] has level θ(Ω^Ω^Ω^Ω^Ω^Ω^θ(1)),
[1 [1 [1 [1 [1 $\backslash_5$ 3] 2] 2] 2] 2] has level θ(Ω^^7),
[1 [1 [1 [1 [1 $\backslash_5$ 1 $\backslash_5$ 2] 2] 2] 2] 2] has level θ(Ω^^8),
[1 [1 [1 [1 [1 [1 $\backslash_6$ 3] 2] 2] 2] 2] 2] has level θ(Ω^^9),
[1 [1 [1 [1 [1 [1 $\backslash_6$ 1 $\backslash_6$ 2] 2] 2] 2] 2] 2] has level θ(Ω^^10).
The sequence of separators starting with the last four has limit ordinal
θ($\varepsilon_{\Omega+1}$) = θ(Ω^^ω) = θ(Ω^Ω^Ω^...) (with ω Ω's),
which is the Bachmann-Howard ordinal. The '+1' is due to the fact that Ω = ε_{Ω} and $\varepsilon_{\Omega+1}$ = ε_{Ω}^^ω.

It can be seen that each additional nested layer of the separator (and increment of the backslash subscript) increases the height of the Ω power tower (within the θ function) by two, with the (n-1)-separator [1 $\backslash_n$ 1 $\backslash_n$ 2] being a halfway house (in hyperlog to base Ω terms) between [1 $\backslash_n$ 3] and [1 [1 $\backslash_{n+1}$ 3] 2].

In general, with n layers of square brackets (n ≥ 2),
[1 [1 [... [1 [1 $\backslash_n$ 3] 2] ...] 2] 2] has level θ(Ω^^(2n-3)),
[1 [1 [... [1 [1 $\backslash_n$ 1 $\backslash_n$ 2] 2] ...] 2] 2] has level θ(Ω^^(2n-2)).
The first of these two separators marks the beginning of level n-1 of my Nested Hyper-Nested Array Notation, while the second separator marks the midway point between levels n-1 and n.

The nesting separator
[1 [1 ... [1 [1 $\backslash_k$ 3] 2] ... 2] 2] (with k-1 layers of brackets, k ≥ 2)
is the lowest special hyperseparator that is a (k-1)-nesting separator – one that requires k-1 pairs of brackets per iteration of the R_n function.

For k ≥ 2, with k layers of square brackets,

{a, b [1 [1 [... [1 [n+1 $\backslash_k$ #] 2] ...] 2] 2] 2} = {a ‹0 [1 [... [1 [n+1 $\backslash_k$ #] 2] ...] 2] 2› b}
= {a ‹b ‹b ‹ ... ‹b ‹n $\backslash_k$ #› b› ... › b› b› b},

where ‘b ‹n $\backslash_k$ #› b’ = ‘b ‹n-1 $\backslash_k$ #› b [n $\backslash_k$ #] b ‹n-1 $\backslash_k$ #› b [n $\backslash_k$ #] ... [n $\backslash_k$ #] b ‹n-1 $\backslash_k$ #› b’
(with b ‘b ‹n-1 $\backslash_k$ #› b’ strings),

‘b ‹1 $\backslash_k$ #› b’ = ‘b $\backslash_{k-1}$ b $\backslash_{k-1}$ b $\backslash_{k-1}$ ... $\backslash_{k-1}$ b’ (with b b’s),

‘b ‹0 $\backslash_k$ #› b’ = ‘b’.

For k ≥ 2, m ≥ 3, with k layers of square brackets,

{a, b [1 [1 [... [1 [1 $\backslash_k$ m] 2] ...] 2] 2] 2} = {a ‹0 [1 [... [1 [1 $\backslash_k$ m] 2] ...] 2] 2› b}
= {a ‹b ‹R_b› b› b},

where R_n = ‘b ‹b ‹ ... ‹b ‹b ‹R_{n-1}› b $\backslash_k$ m-1› b› ... › b› b’ (with k-1 layers of angle brackets),

R_1 = ‘0’.

If k = 2, the above would be

{a, b [1 [1 $\backslash_2$ m] 2] 2} = {a ‹0 [1 $\backslash_2$ m] 2› b}
= {a ‹b ‹R_b› b› b},

where R_n = ‘b ‹R_{n-1}› b $\backslash_2$ m-1’,

R_1 = ‘0’.

For generalised Nested Hyper-Nested Arrays where the number of levels has limit ordinal ω, an extra (middle) subscript has been introduced to the separator arrays in Rule A5 so that the A_{i,i^*}, B_{i,i^*}, C_{i,i^*} etc. strings are written as $A_{i,1,i^*}$, $A_{i,2,i^*}$, $A_{i,3,i^*}$ etc. The first subscript (i) of A_{i,j,i^*} denotes the ith branch, while the second subscript (j) denotes the jth layer of the [$A_{i,1,p_{i,1}}$] separator, where each of the [A_{i,j,i^*}] is a j-hyperseparator (can be a normal separator when j = 1).

Angle Bracket Rule A5 is modified as follows:-

Rule A5 (Rules A1-4 do not apply, first entry is 0, separator immediately prior to next non-1 entry ($c_{1,1}$) is [$A_{1,1,p_{1,1}}$]):

‘a ‹ 0 [$A_{1,1,1}$] 1 [$A_{1,1,2}$] ... 1 [$A_{1,1,p_{1,1}}$] $c_{1,1}$ $\#_{1,1}$ #* › b’ = ‘a ‹$S_{1,1}$ #*› b’,

where $p_{1,1}$ ≥ 1, each of [$A_{1,1,i^*}$] is either normal separator or 1-hyperseparator, $\#_{1,1}$ contains no 2- or higher order hyperseparators in its base layer and #* is either an empty string or begins with a 2- or higher order hyperseparator.
Set i = 1 and follow Rules A5a-e (b-e are terminal, a is not).

Rule A5a (separator

[$A_{i,j,p_{i,j}}$] = [1 [$A_{i,j+1,1}$] 1 [$A_{i,j+1,2}$] ... 1 [$A_{i,j+1,p_{i,j+1}}$] $c_{i,j+1}$ $\#_{i,j+1}$] (1 ≤ j < k)
= [1 [$A_{i,j+1,1}$] 1 [$A_{i,j+1,2}$] ... 1 [$A_{i,j+1,p_{i,j+1}}$]
1 [$A_{i+1,1,1}$] 1 [$A_{i+1,1,2}$] ... 1 [$A_{i+1,1,p_{i+1,1}}$] $c_{i+1,1}$ $\#_{i+1,1}$] (j = k),

where $p_{i,j+1}$ ≥ 1, $p_{i+1,1}$ ≥ 1, $c_{i+1,1}$ ≥ 2, each of [$A_{i+1,1,i^*}$] is either a normal separator or 1-hyperseparator, and each of [$A_{i,j+1,i^*}$] is a (j+1)-hyperseparator):

$S_{i,j}$ = ‘b ‹$A_{i,j,1}$’› b [$A_{i,j,1}$] b ‹$A_{i,j,2}$’› b [$A_{i,j,2}$] ... b ‹$A_{i,j,p_{i,j}-1}$’› b [$A_{i,j,p_{i,j}-1}$]
b ‹$S_{i,j+1}$› b [$A_{i,j,p_{i,j}}$] $c_{i,j}$-1 $\#_{i,j}$’ (1 ≤ j < k+1)
= ‘b ‹$A_{i,j,1}$’› b [$A_{i,j,1}$] b ‹$A_{i,j,2}$’› b [$A_{i,j,2}$] ... b ‹$A_{i,j,p_{i,j}}$’› b [$A_{i,j,p_{i,j}}$] $S_{i+1,1}$’ (j = k+1).

Increment i by 1 and repeat Rules A5a-e.

Rule A5b (separator $[A_{i,1,p_{i,1}}] = [1 \backslash_2 2] = \backslash$):

$S_{i,1} = ‘R_b’$,

$R_n = ‘b \langle A_{i,1,1}' \rangle\ b\ [A_{i,1,1}]\ b \langle A_{i,1,2}' \rangle\ b\ [A_{i,1,2}] \ldots b \langle A_{i,1,p_{i,1}-1}' \rangle\ b\ [A_{i,1,p_{i,1}-1}]$
$b \langle R_{n-1} \rangle\ b\ \backslash\ c_{i,1}-1\ \#_{i,1}’$ $(n > 1)$,

$R_1 = ‘0’$.

Rule A5c (Rules A5a-b do not apply, separator

$[A_{i,j,p_{i,j}}] = [1\ [A_{i,j+1,1}]\ 1\ [A_{i,j+1,2}] \ldots 1\ [A_{i,j+1,p_{i,j+1}}]\ c_{i,j+1}\ \#_{i,j+1}]$ $(1 \le j < k)$
$= [1 \backslash_{j+1} 2] = \backslash_j$ $(j = k)$,

where $p_{i,j+1} \ge 1$, $c_{i,j+1} \ge 2$ and each of $[A_{i,j+1,i^*}]$ is a (j+1)-hyperseparator):

$S_{i,1} = ‘b \langle A_{i,1,1}' \rangle\ b\ [A_{i,1,1}]\ b \langle A_{i,1,2}' \rangle\ b\ [A_{i,1,2}] \ldots b \langle A_{i,1,p_{i,1}-1}' \rangle\ b\ [A_{i,1,p_{i,1}-1}]$
$b \langle R_{b,1} \rangle\ b\ [A_{i,1,p_{i,1}}]\ c_{i,1}-1\ \#_{i,1}’$,

$R_{n,j} = ‘b \langle A_{i,j+1,1}' \rangle\ b\ [A_{i,j+1,1}]\ b \langle A_{i,j+1,2}' \rangle\ b\ [A_{i,j+1,2}] \ldots$
$b \langle A_{i,j+1,p_{i,j+1}-1}' \rangle\ b\ [A_{i,j+1,p_{i,j+1}-1}]\ b \langle R_{n,j+1} \rangle\ b\ [A_{i,j+1,p_{i,j+1}}]\ c_{i,j+1}-1\ \#_{i,j+1}’$ $(n > 1, 1 \le j < k-1)$
$= ‘b \langle A_{i,j+1,1}' \rangle\ b\ [A_{i,j+1,1}]\ b \langle A_{i,j+1,2}' \rangle\ b\ [A_{i,j+1,2}] \ldots b \langle A_{i,j+1,p_{i,j+1}-1}' \rangle\ b\ [A_{i,j+1,p_{i,j+1}-1}]$
$b \langle A_{i,1,1}' \rangle\ b\ [A_{i,1,1}]\ b \langle A_{i,1,2}' \rangle\ b\ [A_{i,1,2}] \ldots b \langle A_{i,1,p_{i,1}-1}' \rangle\ b\ [A_{i,1,p_{i,1}-1}]$
$b \langle R_{n-1,1} \rangle\ b\ [A_{i,1,p_{i,1}}]\ c_{i,1}-1\ \#_{i,1}\ \backslash_{j+1}\ c_{i,j+1}-1\ \#_{i,j+1}’$ $(n > 1, j = k-1)$,

$R_{1,1} = ‘0’$.

Rule A5d (Rules A5a-c do not apply, separator

$[A_{i,j,p_{i,j}}] = [1\ [A_{i,j+1,1}]\ 1\ [A_{i,j+1,2}] \ldots 1\ [A_{i,j+1,p_{i,j+1}}]\ c_{i,j+1}\ \#_{i,j+1}]$ $(1 \le j < k)$
$= [X\ [A_{i,j+1,1}]\ Y]$, $X \ne ‘1’$, $Y \ne ‘1’$ $(j = k \ge 2)$,

where $p_{i,j+1} \ge 1$, $c_{i,j+1} \ge 2$, each of $[A_{i,j+1,i^*}]$ is a (j+1)-hyperseparator and X does not contain any 2- or higher order hyperseparators in its ‘base layer’):

$S_{i,j} = ‘b \langle A_{i,j,1}' \rangle\ b\ [A_{i,j,1}]\ b \langle A_{i,j,2}' \rangle\ b\ [A_{i,j,2}] \ldots b \langle A_{i,j,p_{i,j}-1}' \rangle\ b\ [A_{i,j,p_{i,j}-1}]$
$b \langle S_{i,j+1} \rangle\ b\ [A_{i,j,p_{i,j}}]\ c_{i,j}-1\ \#_{i,j}’$ $(1 \le j < k)$
$= ‘b \langle A_{i,j,1}' \rangle\ b\ [A_{i,j,1}]\ b \langle A_{i,j,2}' \rangle\ b\ [A_{i,j,2}] \ldots b \langle A_{i,j,p_{i,j}}' \rangle\ b\ [A_{i,j,p_{i,j}}]\ c_{i,j}-1\ \#_{i,j}’$ $(j = k)$.

Rule A5e (Rules A5a-d do not apply):

$S_{i,1} = ‘b \langle A_{i,1,1}' \rangle\ b\ [A_{i,1,1}]\ b \langle A_{i,1,2}' \rangle\ b\ [A_{i,1,2}] \ldots b \langle A_{i,1,p_{i,1}}' \rangle\ b\ [A_{i,1,p_{i,1}}]\ c_{i,1}-1\ \#_{i,1}’$.

Rules A5a, A5c and A5d process k-branching, (k-1)-nesting and k-plugging separators respectively (for a given value of k). These rules can be simplified by considering each individual layer separately (as well as each individual branch), and iterating j in the same manner as i. The possible base-layer scenarios of the $[A_{i,j,p_{i,j}}]$ separator fall into five sets or categories (each to be served by different subrules), these are as follows:-

Set 1: $[1 \backslash_2 2] = \backslash$ $(j = 1)$.

Set 2: $[1 \backslash_{j+1} 2] = \backslash_j$ $(j \ge 2)$.

Set 3: $[1\ [A_{i,j+1,1}]\ 1\ [A_{i,j+1,2}] \ldots 1\ [A_{i,j+1,p_{i,j+1}}]\ 1\ [A_{i+1,1,1}]\ 1\ [A_{i+1,1,2}] \ldots 1\ [A_{i+1,1,p_{i+1,1}}]\ c_{i+1,1}\ \#_{i+1,1}]$.

Set 4: $[1\ [A_{i,j+1,1}]\ 1\ [A_{i,j+1,2}] \ldots 1\ [A_{i,j+1,p_{i,j+1}}]\ c_{i,j+1}\ \#_{i,j+1}]$.

Set 5: All other scenarios.

In Sets 3 and 4, $p_{i,j+1} \ge 1$, each of $[A_{i,j+1,i^*}]$ is a (j+1)-hyperseparator and the # strings represent the rest of the separator array (which may be null strings). Additionally, in Set 3, $p_{i+1,1} \ge 1$, $c_{i+1,1} \ge 2$, and each of $[A_{i+1,1,i^*}]$ is either a normal separator or 1-hyperseparator; in Set 4, $c_{i,j+1} \ge 2$.

Set 1 covers 1-nesting separators (trivial case), Set 2 covers the top layers of nesting separators (other than Set 1), Set 3 covers the top layers of branching separators, Set 4 covers the lower layers of all three types of special separators and Set 5 covers all other (or non-special) separators (including the top layers of plugging separators).

The modified and complete Angle Bracket Rules – including the revised subrules of Rule A5 – are as follows:-

Rule A1 (only 1 entry of either 0 or 1):

‘a ‹0› b’ = ‘a’,

‘a ‹1› b’ = ‘a, a, ... , a’ (with b a’s).

Rule A2 (only 1 entry of either 0 or 1 prior to 2-hyperseparator or higher order hyperseparator):

‘a ‹0 #› b’ = ‘a’,

‘a ‹1 #› b’ = ‘a [1 #] a [1 #] ... [1 #] a’ (with b a’s),

where # begins with a 2- or higher order hyperseparator.

When n ≥ 2,

‘a ‹1 $\backslash_n$ 2› b’ = ‘a $\backslash_{n-1}$ a $\backslash_{n-1}$ ... $\backslash_{n-1}$ a’ (with b a’s).

Rule A3 (last entry in any 1-space or higher dimensional space of array is 1):

‘a ‹# [A] 1› b’ = ‘a ‹#› b’.

When [A] is an m-hyperseparator, [B] is an n-hyperseparator and m < n, or m = n and level of [A] is less than level of [B],

‘a ‹# [A] 1 [B] #*› b’ = ‘a ‹# [B] #*› b’.

Remove trailing 1’s.

Rule A4 (number to right of angle brackets is 1):

‘a ‹A› 1’ = ‘a’.

Rule A5 (Rules A1-4 do not apply, first entry is 0, separator immediately prior to next non-1 entry ($c_{1,1}$) is [$A_{1,1,p_{1,1}}$]):

‘a ‹ 0 [$A_{1,1,1}$] 1 [$A_{1,1,2}$] ... 1 [$A_{1,1,p_{1,1}}$] $c_{1,1}$ $\#_{1,1}$ #* › b’ = ‘a ‹$S_{1,1}$ #*› b’,

where $p_{1,1}$ ≥ 1, each of [$A_{1,1,i^*}$] is either normal separator or 1-hyperseparator, $\#_{1,1}$ contains no 2- or higher order hyperseparators in its base layer and #* is either an empty string or begins with a 2- or higher order hyperseparator.

Set i = 1 and j = 1, and follow Rules A5a-e (a, b and e are terminal, c and d are not).

Rule A5a (separator [$A_{i,1,p_{i,1}}$] = [1 $\backslash_2$ 2] = \):

$S_{i,1}$ = ‘R_b’,

R_n = ‘b ‹$A_{i,1,1}$’› b [$A_{i,1,1}$] b ‹$A_{i,1,2}$’› b [$A_{i,1,2}$] ... b ‹$A_{i,1,p_{i,1}-1}$’› b [$A_{i,1,p_{i,1}-1}$]
b ‹R_{n-1}› b \ $c_{i,1}$-1 $\#_{i,1}$’ (n > 1),

R_1 = ‘0’.

Rule A5b (Rule A5a does not apply, separator [$A_{i,j,p_{i,j}}$] = [1 $\backslash_{j+1}$ 2] = $\backslash_j$, where j ≥ 2):

$S_{i,j}$ = ‘$R_{b,j-1}$’,

$R_{n,j-1}$ = ‘b ‹$A_{i,j,1}$’› b [$A_{i,j,1}$] b ‹$A_{i,j,2}$’› b [$A_{i,j,2}$] ... b ‹$A_{i,j,p_{i,j}-1}$’› b [$A_{i,j,p_{i,j}-1}$]
b ‹$A_{i,1,1}$’› b [$A_{i,1,1}$] b ‹$A_{i,1,2}$’› b [$A_{i,1,2}$] ... b ‹$A_{i,1,p_{i,1}-1}$’› b [$A_{i,1,p_{i,1}-1}$]
b ‹$R_{n-1,1}$› b [$A_{i,1,p_{i,1}}$] $c_{i,1}$-1 $\#_{i,1}$ $\backslash_j$ $c_{i,j}$-1 $\#_{i,j}$’ (n > 1),

$R_{n,k}$ = ‘b ‹$A_{i,k+1,1}$’› b [$A_{i,k+1,1}$] b ‹$A_{i,k+1,2}$’› b [$A_{i,k+1,2}$] ...
b ‹$A_{i,k+1,p_{i,k+1}-1}$’› b [$A_{i,k+1,p_{i,k+1}-1}$] b ‹$R_{n,k+1}$› b [$A_{i,k+1,p_{i,k+1}}$] $c_{i,k+1}$-1 $\#_{i,k+1}$’ (n > 1, 1 ≤ k < j-1),

$R_{1,1}$ = ‘0’.

Rule A5c (Rules A5a-b do not apply, separator

$[A_{i,j,p_{i,j}}] = [1\ [A_{i,j+1,1}]\ 1\ [A_{i,j+1,2}]\ \ldots\ 1\ [A_{i,j+1,p_{i,j+1}}]\ 1\ [A_{i+1,1,1}]\ 1\ [A_{i+1,1,2}]\ \ldots\ 1\ [A_{i+1,1,p_{i+1,1}}]\ c_{i+1,1}\ \#_{i+1,1}]$,

where $p_{i,j+1} \geq 1$, $p_{i+1,1} \geq 1$, $c_{i+1,1} \geq 2$, each of $[A_{i+1,1,i^*}]$ is either a normal separator or 1-hyperseparator, and each of $[A_{i,j+1,i^*}]$ is a (j+1)-hyperseparator):

$S_{i,j}$ = 'b $\langle A_{i,j,1}' \rangle$ b $[A_{i,j,1}]$ b $\langle A_{i,j,2}' \rangle$ b $[A_{i,j,2}]$ ... b $\langle A_{i,j,p_{i,j}-1}' \rangle$ b $[A_{i,j,p_{i,j}-1}]$ b $\langle T_i \rangle$ b $[A_{i,j,p_{i,j}}]$ $c_{i,j}$-1 $\#_{i,j}$',

T_i = 'b $\langle A_{i,j+1,1}' \rangle$ b $[A_{i,j+1,1}]$ b $\langle A_{i,j+1,2}' \rangle$ b $[A_{i,j+1,2}]$ ... b $\langle A_{i,j+1,p_{i,j+1}}' \rangle$ b $[A_{i,j+1,p_{i,j+1}}]$ $S_{i+1,1}$'.

Increment i by 1, reset j = 1, and repeat Rules A5a-e.

Rule A5d (Rules A5a-c do not apply, separator

$[A_{i,j,p_{i,j}}] = [1\ [A_{i,j+1,1}]\ 1\ [A_{i,j+1,2}]\ \ldots\ 1\ [A_{i,j+1,p_{i,j+1}}]\ c_{i,j+1}\ \#_{i,j+1}]$,

where $p_{i,j+1} \geq 1$, $c_{i,j+1} \geq 2$ and each of $[A_{i,j+1,i^*}]$ is a (j+1)-hyperseparator):

$S_{i,j}$ = 'b $\langle A_{i,j,1}' \rangle$ b $[A_{i,j,1}]$ b $\langle A_{i,j,2}' \rangle$ b $[A_{i,j,2}]$ ... b $\langle A_{i,j,p_{i,j}-1}' \rangle$ b $[A_{i,j,p_{i,j}-1}]$ b $\langle S_{i,j+1} \rangle$ b $[A_{i,j,p_{i,j}}]$ $c_{i,j}$-1 $\#_{i,j}$'.

Increment j by 1 and repeat Rules A5a-e.

Rule A5e (Rules A5a-d do not apply):

$S_{i,j}$ = 'b $\langle A_{i,j,1}' \rangle$ b $[A_{i,j,1}]$ b $\langle A_{i,j,2}' \rangle$ b $[A_{i,j,2}]$ ... b $\langle A_{i,j,p_{i,j}}' \rangle$ b $[A_{i,j,p_{i,j}}]$ $c_{i,j}$-1 $\#_{i,j}$'.

Rule A6 (Rules A1-5 do not apply):

'a ‹n #› b' = 'a ‹n-1 #› b [n #] a ‹n-1 #› b [n #] ... [n #] a ‹n-1 #› b'
(with b 'a ‹n-1 #› b' strings).

Notes:

1. A, B, $A_{i,j,1}$, $A_{i,j,2}$, ... , $A_{i,j,p_{i,j}}$ are strings of characters within separators.
2. $A_{i,j,1}'$, $A_{i,j,2}'$, ... , $A_{i,j,p_{i,j}}'$ are strings of characters within angle brackets that are identical to the strings $A_{i,j,1}$, $A_{i,j,2}$, ... , $A_{i,j,p_{i,j}}$ respectively except that the first entries of each have been reduced by 1. If A_{i,j,i^*} (for some $1 \leq i^* \leq p_{i,j}$) begins with 1, A_{i,j,i^*}' begins with 0.
3. $S_{i,j}$, T_i, R_n and $R_{n,k}$ are string building functions which create strings of characters. The R functions involve nesting the same string of characters around itself n-1 times before being replaced by the string '0'.
4. #, #* and $\#_{i,j}$ are strings of characters representing the remainder of the array (can be null or empty).
5. A $\backslash_n$ symbol is an n-hyperseparator, $\backslash_n$ enclosed by m pairs of square brackets is an (n-m)-hyperseparator (when n > m) and a normal separator (or 0-hyperseparator) otherwise. A separator containing no backslashes whatsoever is a normal separator.
6. The comma is used as shorthand for the [1] separator.
7. $\backslash_n$ is used as shorthand for the [1 $\backslash_{n+1}$ 2] separator.

This is my complete Nested Hyper-Nested Array Notation. The limit ordinal of this notation is $\theta(\varepsilon_{\Omega+1})$, the Bachmann-Howard ordinal. This notation works for simple nested arrays up to ε_0 level without requiring Rules A2 and A5a-d. Rule A5a is required to reach ε_0 (represented by the [1 \ 2] separator), Rule A2 is needed to achieve φ(ω, 0) (or use the [1 [2 $\backslash_2$ 2] 2] separator), Rules A5b and A5d are required to get to Γ_0 (or process [1 [1 $\backslash_2$ 3] 2] or beyond) and Rule A5c was brought in to gain access to θ(Ω^ω) or the small Veblen ordinal (or deal with [1 [1 $\backslash_2$ 1,2] 2] upwards).

An n-hyperseparator (n ≥ 1) is of the form

$[A_1\ [X_1]\ A_2\ [X_2]\ \ldots\ [X_k]\ A_{k+1}]$,

where k ≥ 1, each of $[X_i]$ is an (n+1)-hyperseparator (with the $\backslash_{n+1}$ symbol used in place of [1 $\backslash_{n+2}$ 2]) and each of the A_i strings within square brackets would be a normal separator. In Rule A3, the levels of two n-hyperseparators [A] and [B] are determined by the highest ranking (n+1)-hyperseparator within their ‘base layers’, then the numbers of them when they are identical. When the numbers are equal, this is repeated for the subarrays of [A] and [B] to the right of the rightmost highest ranking (n+1)-hyperseparator. When the highest ranking (n+1)-hyperseparators and their numbers within the subarrays are identical, this is repeated again for the subarrays within the subarrays, until no more (n+1)-hyperseparators remain, in which case the ordinal levels of the two subarrays (normal separators when placed within square brackets) are then considered (see page 22 of Beyond Bird’s Nested Arrays I for details on how the levels of two normal separators are determined). If the ordinal levels are the same, the ‘[X_k] A_{k+1}’ strings (from the final (n+1)-hyperseparator onwards) from each of [A] and [B] are deleted, and the entire process is repeated for the truncated [A] and [B], until these become normal separators (in which case their ordinal levels are taken into account). When we get a lower level or number for [A] than for [B] on some measure, then the original [A] ranks lower than the original [B] and the ‘[A] 1’ string is deleted. (The levels of (n+1)-hyperseparators are first determined by the highest ranking (n+2)-hyperseparator within their ‘base layers’, and so on.)

In Rule A5b, where I have written “separator [$A_{i,j,p_{i,j}}$] = [1 $\backslash_{j+1}$ 2] = $\backslash_j$, where j ≥ 2”, the $\backslash_j$ has been written as a shorthand for [1 $\backslash_{j+1}$ 2]. I have included the [1 $\backslash_{j+1}$ 2] since $A_{i,j,p_{i,j}}$ is a string within square brackets, beginning with a number. There may exist [$A_{i,j,i*}$] = [1 $\backslash_{j+1}$ 2] for some i* < $p_{i,j}$, in which case we need to find ‘b ‹$A_{i,j,i*}$’› b’, which would be ‘b ‹0 $\backslash_{j+1}$ 2› b’ = ‘b’ (as j+1 ≥ 2). Some form of shorthand is necessary since there are an infinite number of ways of writing $\backslash_n$, namely

[1 $\backslash_{n+1}$ 2], [1 [1 $\backslash_{n+2}$ 2] 2], [1 [1 [1 $\backslash_{n+3}$ 2] 2] 2], ...

The following function grows so rapidly that its growth rate is of the magnitude of the Bachmann-Howard ordinal, which is probably the largest ordinal with a special name.

H(n) = {3, n [1 [1 [... [1 [1 $\backslash_n$ 1 $\backslash_n$ 2] 2] ...] 2] 2] 2} (with n layers of square brackets).

The first four values of the H function are as follows:

H(1) = 3,
H(2) = {3, 2 [1 [1 $\backslash_2$ 1 $\backslash_2$ 2] 2] 2}
= {3 ‹0 [1 $\backslash_2$ 1 $\backslash_2$ 2] 2› 2}
= {3 ‹2 ‹2 $\backslash_2$ 2› 2› 2}
= {3 ‹2 \ 2 [2 $\backslash_2$ 2] 2 \ 2› 2},
H(3) = {3, 3 [1 [1 [1 $\backslash_3$ 1 $\backslash_3$ 2] 2] 2] 2}
= {3 ‹0 [1 [1 $\backslash_3$ 1 $\backslash_3$ 2] 2] 2› 3}
= {3 ‹3 ‹3 ‹3 $\backslash_3$ 3 ‹3 ‹3 $\backslash_3$ 3› 3› 3› 3› 3› 3},
H(4) = {3, 4 [1 [1 [1 [1 $\backslash_4$ 1 $\backslash_4$ 2] 2] 2] 2] 2}
= {3 ‹0 [1 [1 [1 $\backslash_4$ 1 $\backslash_4$ 2] 2] 2] 2› 4}
= {3 ‹4 ‹4 ‹4 ‹4 $\backslash_4$ 4 ‹4 ‹4 ‹4 $\backslash_4$ 4 ‹4 ‹4 ‹4 $\backslash_4$ 4› 4› 4› 4› 4› 4› 4› 4› 4› 4› 4}.

Imagine how huge this number must be:

N = H(H(H(...H(3)...))) (with H(3) H’s).

Is this a record for the largest finite number defined or created by a notation? Imagine how far away is N light years if the universe is infinite! Imagine a group of N galaxies in our infinite universe, with some of those galaxies large enough to contain at least N stars, N planets and even N black holes. Imagine a black hole that is N light years in diameter. And that’s not to mention multiverses or hyperuniverses with N or an infinite number of universes, 2-multiverses with N or an infinite number of multiverses, and even N-multiverses!

Author: Chris Bird (Gloucestershire, England, UK)
Last modified: 14 March 2014
E-mail: m.bird44 at btinternet.com (not clickable to thwart spambots!)