Beyond Bird's Nested Arrays II

The most significant separators introduced in the first part are:-

[1] has level 0,
[2] has level 1,
[1, 2] has level ω,
[1 [2] 2] has level ω^ω,
[1 [1, 2] 2] has level ω^ω^ω,
[1 [1 [2] 2] 2] has level ω^ω^ω^ω,
[1 \ 2] has level ε_0,
[1 [1 \ 2] 2 \ 2] has level $\varepsilon_0 2$,
[1 [1 [1 \ 2] 2 \ 2] 2 \ 2] has level ε_0^ε_0,
[1 \ 3] has level ε_1,
[1 \ 4] has level ε_2,
[1 \ 1, 2] has level ε_ω,
[1 \ 1 [1 \ 2] 2] has level $\varepsilon(\varepsilon_0)$,
[1 \ 1 \ 2] has level $\zeta_0 = \varphi(2, 0)$,
[1 \ 1 \ 1 \ 2] has level φ(3, 0),
[1 [2]\ 2] has level φ(ω, 0),
[1 [1, 2]\ 2] has level φ(ω^ω, 0),
[1 [1 [2] 2]\ 2] has level φ(ω^ω^ω, 0),
[1 [1 [1, 2] 2]\ 2] has level φ(ω^ω^ω^ω, 0),
[1 [1 \ 2]\ 2] has level $\varphi(\varepsilon_0, 0)$,
[1 [1 [1 \ 2]\ 2]\ 2] has level $\varphi(\varphi(\varepsilon_0, 0), 0)$,
[1 [1 [1 [1 \ 2]\ 2]\ 2]\ 2] has level $\varphi(\varphi(\varphi(\varepsilon_0, 0), 0), 0)$,
[1 / 2] has level $\Gamma_0 = \varphi(1, 0, 0)$.

Note: \ is shorthand for [1]\ just as the comma is shorthand for [1].

There are now three hyperlevels of inner separators (enclosed by at least one layer of square brackets). The first hyperlevel (normal separators) comprises those without backslashes appended to them. The second hyperlevel (hyper-2 separators) comprises those with backslashes attached to them. The forward slash denotes the beginning of the third hyperlevel of separators (or hyper-3 separators).

The forward slash can be easily confused with the backslash and I wish to extend my notation. At this stage it may be easier to use the negation symbol (¬) and rewrite the hyper-2 separators [X]\ as [X¬2]. The backslash \ now becomes shorthand for the [1¬2] hyperseparator. This is because when I go down a hyperlevel to the normal separators, the 2 reduces to 1 and the '¬1' is removed (remove trailing 1's along with the marker or separator, as with normal arrays). The forward slash can now be rewritten as [1¬3]. Hyper-2 level arrays within angle brackets '‹X›\' can be rewritten as '‹X¬2›'. Note that the ¬ symbol is so special that there are always a minimum of two square or angle brackets enclosing it, just as there are a minimum of one square or angle brackets enclosing backslashes.

The most significant separators introduced so far (from [1 [2]\ 2] onwards) can now be rewritten as follows:-

[1 [2¬2] 2] has level φ(ω, 0),
[1 [1, 2 ¬2] 2] has level φ(ω^ω, 0),
[1 [1 [2] 2 ¬2] 2] has level φ(ω^ω^ω, 0),
[1 [1 [1, 2] 2 ¬2] 2] has level φ(ω^ω^ω^ω, 0),

[1 [1 \ 2 ¬2] 2] has level φ(ε_0, 0),
[1 [1 [1 \ 2 ¬2] 2 ¬2] 2] has level φ(φ(ε_0, 0), 0),
[1 [1 [1 [1 \ 2 ¬2] 2 ¬2] 2 ¬2] 2] has level φ(φ(φ(ε_0, 0), 0), 0),
[1 [1¬3] 2] has level Γ_0 = φ(1, 0, 0).

Note: \ is now shorthand for [1¬2] just as the comma is shorthand for [1].

The φ(ω^n, 0) level separator

{a, b [1 [n+1 ¬2] 2] 2} = {a ‹0 [n+1 ¬2] 2› b}
= {a ‹b ‹n¬2› b› b},

where   ‘b ‹n¬2› b’ = ‘b ‹n-1 ¬2› b [n¬2] b ‹n-1 ¬2› b [n¬2] ... [n¬2] b ‹n-1 ¬2› b’
(with b ‘b ‹n-1 ¬2› b’ strings),
‘b ‹1¬2› b’ = ‘b \ b \ b \ ... \ b’   (with b b’s),
‘b ‹0¬2› b’ = ‘b’.

I can now define the Γ_0 level separator:

{a, b [1 [1¬3] 2] 2} = {a ‹0 [1¬3] 2› b}
(*not* {a ‹b ‹0¬3› b› b} = {a ‹b› b} – special rules apply)
= {a ‹b ‹b ‹b ‹ ... ‹b ‹b¬2› b¬2› ... › b¬2› b¬2› b› b}
(with b+1 b’s from centre to right and b-1 ¬’s)
= {a ‹b ‹b ‹b ‹ ... ‹b ‹b›\ b›\ ... ›\ b›\ b›\ b› b}
(with b+1 b’s from centre to right in previous ‘backslash array’ notation).

This is analogous to the ε_0 level separator, except that the notation ‘drops down’ a level for the latter (the [1¬2] ‘drops down’ to a backslash):

{a, b [1 [1¬2] 2] 2} = {a, b [1 \ 2] 2}
= {a ‹0 \ 2› b}
= {a ‹b ‹b ‹ ... ‹b ‹b \ 1› b \ 1› ... › b \ 1› b \ 1› b}   (b b’s from centre to right)
= {a ‹b ‹b ‹ ... ‹b ‹b› b› ... › b› b› b}   (b b’s from centre to right).

While {a, b [1 \ 2] 2} involves b-1 layers of normal (hyper-1 level) nested angle brackets, {a, b [1 [1¬3] 2] 2} involves b layers of nested angle brackets, of which b-1 are at hyper-2 level – the latter ‘steps up a gear’, in order to accommodate the same number of nested layers at the highest hyperlevel (due to the rules of the notation), which is why the notation ‘drops down’ a level for the smaller array.

Since I have extended my notation to the next level to deal with separators of [1 [2¬2] 2] (φ(ω, 0) level) and above, the Γ_0 level function is larger than before. For example,

{3, 2 [1 [1¬3] 2] 2} = {3 ‹0 [1¬3] 2› 2}
= {3 ‹2 ‹2¬2› 2› 2}
= {3 ‹2 [1¬2] 2 [2¬2] 2 [1¬2] 2› 2}
= {3 ‹2 \ 2 [2¬2] 2 \ 2› 2}   (the [1¬2] ‘drops down’ to \)
= {A [1 \ 2 [2¬2] 2 \ 2] A [2 \ 2 [2¬2] 2 \ 2] A [1 \ 2 [2¬2] 2 \ 2] A},

where   A = ‘3 ‹0 \ 2 [2¬2] 2 \ 2› 2’
= ‘3 ‹2 [2¬2] 2 \ 2› 2’
= ‘B [1 [2¬2] 2 \ 2] B [2 [2¬2] 2 \ 2] B [1 [2¬2] 2 \ 2] B’,

where   B = ‘3 ‹0 [2¬2] 2 \ 2› 2’
= ‘3 ‹2 \ 2 [2¬2] 1 \ 2› 2’
= ‘C [1 \ 2 [2¬2] 1 \ 2] C [2 \ 2 [2¬2] 1 \ 2] C [1 \ 2 [2¬2] 1 \ 2] C’,

where   C = ‘3 ‹0 \ 2 [2¬2] 1 \ 2› 2’

= ‘3 ‹2 [2¬2] 1 \ 2› 2’
= ‘D [1 [2¬2] 1 \ 2] D [2 [2¬2] 1 \ 2] D [1 [2¬2] 1 \ 2] D’,
where D = ‘3 ‹0 [2¬2] 1 \ 2› 2’
= ‘3 ‹2 \ 2 [2¬2] 2› 2’
= ‘E [1 \ 2 [2¬2] 2] E [2 \ 2 [2¬2] 2] E [1 \ 2 [2¬2] 2] E’,
where E = ‘3 ‹0 \ 2 [2¬2] 2› 2’
= ‘3 ‹2 [2¬2] 2› 2’
= ‘F [1 [2¬2] 2] F [2 [2¬2] 2] F [1 [2¬2] 2] F’,
where F = ‘3 ‹0 [2¬2] 2› 2’
= ‘3 ‹2 \ 2› 2’
= ‘3 ‹0\2› 2 [1\2] 3 ‹0\2› 2 [2\2] 3 ‹0\2› 2 [1\2] 3 ‹0\2› 2’
= ‘3,3 [2] 3,3 [1\2] 3,3 [2] 3,3 [2\2] 3,3 [2] 3,3 [1\2] 3,3 [2] 3,3’.

{3, 3 [1 [1¬3] 2] 2} = {3 ‹0 [1¬3] 2› 3}
= {3 ‹3 ‹3 ‹3¬2› 3¬2› 3› 3}
= {3 ‹3 ‹ 3\3\3 [2¬2] 3\3\3 [2¬2] 3\3\3 [3¬2]
3\3\3 [2¬2] 3\3\3 [2¬2] 3\3\3 [3¬2]
3\3\3 [2¬2] 3\3\3 [2¬2] 3\3\3 ¬2› 3› 3},
{3, 4 [1 [1¬3] 2] 2} = {3 ‹0 [1¬3] 2› 4}
= {3 ‹4 ‹4 ‹4 ‹4¬2› 4¬2› 4¬2› 4› 4},
{3, 5 [1 [1¬3] 2] 2} = {3 ‹0 [1¬3] 2› 5}
= {3 ‹5 ‹5 ‹5 ‹5 ‹5¬2› 5¬2› 5¬2› 5¬2› 5› 5}.

Friedman’s TREE sequence (finite form of Kruskal’s Tree Theorem) grows more rapidly than the above sequence of numbers, generated by the function
f(n) = {3, n [1 [1¬3] 2] 2}.

[1 [1¬3] 2] has level Γ_0 = φ(1, 0, 0),
[2 [1¬3] 2] has level Γ_0+1,
[1 [1 \ 2] 2 [1¬3] 2] has level $\Gamma_0+\varepsilon_0$,
[1 [1 [1¬3] 2] 2 [1¬3] 2] has level $\Gamma_0$2,
[1 [1 [1 [1¬3] 2] 2 [1¬3] 2] 2 [1¬3] 2] has level Γ_0^Γ_0,
[1 \ 2 [1¬3] 2] has level ε(Γ_0+1) = Γ_0^^ω,
[1 \ 3 [1¬3] 2] has level ε(Γ_0+2),
[1 \ 1 \ 2 [1¬3] 2] has level ζ(Γ_0+1) = φ(2, Γ_0+1),
[1 [2¬2] 2 [1¬3] 2] has level φ(ω, Γ_0+1),
[1 [3¬2] 2 [1¬3] 2] has level φ(ω^2, Γ_0+1),
[1 [1, 2 ¬2] 2 [1¬3] 2] has level φ(ω^ω, Γ_0+1),
[1 [1 \ 2 ¬2] 2 [1¬3] 2] has level φ(ε_0, Γ_0+1),
[1 [1 [1¬3] 2 ¬2] 2 [1¬3] 2] has level φ(Γ_0, 1)
(limit ordinal of φ(α, Γ_0+1) = φ(α, φ(Γ_0, 0)+1) as α → Γ_0),
[1 [1 [1 [1¬3] 2 ¬2] 2 [1¬3] 2 ¬2] 2 [1¬3] 2] has level φ(φ(Γ_0, 1), 0),
[1 [1 [1 [1 [1¬3] 2 ¬2] 2 [1¬3] 2 ¬2] 2 [1¬3] 2 ¬2] 2 [1¬3] 2] has level φ(φ(φ(Γ_0, 1), 0), 0).
The sequence of separators starting with the last three has limit ordinal Γ_1 = φ(1, 0, 1), since we can let φ(Γ_0, 1) collapse to Γ_0+1.

[1 [1¬3] 3] has level Γ_1,
[1 [1¬3] 4] has level Γ_2,
[1 [1¬3] n] has level Γ_{n-2} = φ(1, 0, n-2),

[1 [1¬3] 1, 2] has level Γ_{ω},
[1 [1¬3] 1 [1 \ 2] 2] has level $\Gamma(\varepsilon_0)$,
[1 [1¬3] 1 [1 [1¬3] 2] 2] has level $\Gamma(\Gamma_0)$,
[1 [1¬3] 1 [1 [1¬3] 1 [1 [1¬3] 2] 2] 2] has level $\Gamma(\Gamma(\Gamma_0))$,
[1 [1¬3] 1 [1 [1¬3] 1 [1 [1¬3] 1 [1 [1¬3] 2] 2] 2] 2] has level $\Gamma(\Gamma(\Gamma(\Gamma_0)))$.

[1 [1¬3] 1 \ 2] has level $\varphi(1, 1, 0) = \Gamma_{\varphi(1, 1, 0)} = \Gamma(\Gamma(...(\Gamma_0)...))$ (ω Γ's),
[1 [1¬3] 2 \ 2] has level $\Gamma_{\varphi(1, 1, 0)+1}$,
[1 [1¬3] 1 \ 3] has level $\varphi(1, 1, 1) = \Gamma(\Gamma(...(\Gamma_{\varphi(1, 1, 0)+1})...))$ (ω Γ's),
[1 [1¬3] 1 \ 4] has level $\varphi(1, 1, 2)$,
[1 [1¬3] 1 \ 1 [1 [1¬3] 1 \ 2] 2] has level $\varphi(1, 1, \varphi(1, 1, 0))$,
[1 [1¬3] 1 \ 1 \ 2] has level $\varphi(1, 2, 0)$,
[1 [1¬3] 1 \ 1 \ 1 \ 2] has level $\varphi(1, 3, 0)$,
[1 [1¬3] 1 [2¬2] 2] has level $\varphi(1, \omega, 0)$,
[1 [1¬3] 1 [3¬2] 2] has level $\varphi(1, \omega\text{^}2, 0)$,
[1 [1¬3] 1 [1, 2 ¬2] 2] has level $\varphi(1, \omega\text{^}\omega, 0)$,
[1 [1¬3] 1 [1 \ 2 ¬2] 2] has level $\varphi(1, \varepsilon_0, 0)$,
[1 [1¬3] 1 [1 [1¬3] 2 ¬2] 2] has level $\varphi(1, \Gamma_0, 0)$,
[1 [1¬3] 1 [1 [1¬3] 1 [1 [1¬3] 2 ¬2] 2 ¬2] 2] has level $\varphi(1, \varphi(1, \Gamma_0, 0), 0)$,
[1 [1¬3] 1 [1 [1¬3] 1 [1 [1¬3] 1 [1 [1¬3] 2 ¬2] 2 ¬2] 2 ¬2] 2] has level
$\varphi(1, \varphi(1, \varphi(1, \Gamma_0, 0), 0), 0)$.

[1 [1¬3] 1 [1¬3] 2] has level $\varphi(2, 0, 0)$,
[1 [1¬3] 2 [1¬3] 2] has level $\Gamma_{\varphi(2, 0, 0)+1} = \Gamma_{\varphi(1, 1, \varphi(2, 0, 0))+1}$,
[1 [1¬3] 1 \ 2 [1¬3] 2] has level $\varphi(1, 1, \varphi(2, 0, 0)+1)$
$= \Gamma(\Gamma(...(\Gamma_{\varphi(1, 1, \varphi(2, 0, 0))+1})...))$ (ω Γ's),
[1 [1¬3] 1 \ 1 \ 2 [1¬3] 2] has level $\varphi(1, 2, \varphi(2, 0, 0)+1)$,
[1 [1¬3] 1 [2¬2] 2 [1¬3] 2] has level $\varphi(1, \omega, \varphi(2, 0, 0)+1)$,
[1 [1¬3] 1 [1 \ 2 ¬2] 2 [1¬3] 2] has level $\varphi(1, \varepsilon_0, \varphi(2, 0, 0)+1)$,
[1 [1¬3] 1 [1 [1¬3] 2 ¬2] 2 [1¬3] 2] has level $\varphi(1, \Gamma_0, \varphi(2, 0, 0)+1)$,
[1 [1¬3] 1 [1 [1¬3] 1 [1¬3] 2 ¬2] 2 [1¬3] 2] has level $\varphi(1, \varphi(2, 0, 0), 1)$
(limit ordinal of $\varphi(1, \alpha, \varphi(2, 0, 0)+1) = \varphi(1, \alpha, \varphi(1, \varphi(2, 0, 0), 0)+1)$ as $\alpha \to \varphi(2, 0, 0)$),
[1 [1¬3] 1 [1 [1¬3] 1 [1 [1¬3] 1 [1¬3] 2 ¬2] 2 [1¬3] 2 ¬2] 2 [1¬3] 2] has level
$\varphi(1, \varphi(1, \varphi(2, 0, 0), 1), 0)$,
[1 [1¬3] 1 [1 [1¬3] 1 [1 [1¬3] 1 [1 [1¬3] 1 [1¬3] 2 ¬2] 2 [1¬3] 2 ¬2] 2 [1¬3] 2 ¬2] 2 [1¬3] 2]
has level $\varphi(1, \varphi(1, \varphi(1, \varphi(2, 0, 0), 1), 0), 0)$,
[1 [1¬3] 1 [1¬3] 3] has level $\varphi(2, 0, 1)$,
[1 [1¬3] 1 [1¬3] 4] has level $\varphi(2, 0, 2)$,
[1 [1¬3] 1 [1¬3] 1 [1 [1¬3] 1 [1¬3] 2] 2] has level $\varphi(2, 0, \varphi(2, 0, 0))$,
[1 [1¬3] 1 [1¬3] 1 \ 2] has level $\varphi(2, 1, 0)$,
[1 [1¬3] 1 [1¬3] 1 \ 3] has level $\varphi(2, 1, 1)$,
[1 [1¬3] 1 [1¬3] 1 \ 1 \ 2] has level $\varphi(2, 2, 0)$,
[1 [1¬3] 1 [1¬3] 1 \ 1 \ 1 \ 2] has level $\varphi(2, 3, 0)$,
[1 [1¬3] 1 [1¬3] 1 [2¬2] 2] has level $\varphi(2, \omega, 0)$,
[1 [1¬3] 1 [1¬3] 1 [1 [1¬3] 1 [1¬3] 2 ¬2] 2] has level $\varphi(2, \varphi(2, 0, 0), 0)$,
[1 [1¬3] 1 [1¬3] 1 [1 [1¬3] 1 [1¬3] 1 [1 [1¬3] 1 [1¬3] 2 ¬2] 2 ¬2] 2] has level
$\varphi(2, \varphi(2, \varphi(2, 0, 0), 0), 0)$.

[1 [1¬3] 1 [1¬3] 1 [1¬3] 2] has level $\varphi(3, 0, 0)$,

[1 [1¬3] 1 [1¬3] 1 [1¬3] 1 [1¬3] 2] has level φ(4, 0, 0),
[1 [1¬3] 1 [1¬3] ... [1¬3] 1 [1¬3] 2] (with n [1¬3]'s) has level φ(n, 0, 0),
[1 [1¬3] 1 [1¬3] ... [1¬3] 1 [1¬3] 1 \ 1 \ ... \ 1 \ k] (with n [1¬3]'s, m \'s) has level φ(n, m, k-2).

[1 [2¬3] 2] has level φ(ω, 0, 0),
[1 \ 2 [2¬3] 2] has level $\varepsilon_{\varphi(\omega, 0, 0)+1}$,
[1 [1¬3] 2 [2¬3] 2] has level $\Gamma_{\varphi(\omega, 0, 0)+1}$,
[1 [1¬3] 1 [1¬3] 2 [2¬3] 2] has level φ(2, 0, φ(ω, 0, 0)+1),
[1 [1¬3] 1 [1¬3] ... [1¬3] 1 [1¬3] 2 [2¬3] 2] (with n [1¬3]'s) has level φ(n, 0, φ(ω, 0, 0)+1),
[1 [2¬3] 3] has level φ(ω, 0, 1) (limit ordinal of φ(n, 0, φ(ω, 0, 0)+1) as n → ω),
[1 [2¬3] 4] has level φ(ω, 0, 2),
[1 [2¬3] 1, 2] has level φ(ω, 0, ω),
[1 [2¬3] 1 [1 [2¬3] 2] 2] has level φ(ω, 0, φ(ω, 0, 0)),
[1 [2¬3] 1 [1 [2¬3] 1 [1 [2¬3] 2] 2] 2] has level φ(ω, 0, φ(ω, 0, φ(ω, 0, 0))),
[1 [2¬3] 1 \ 2] has level φ(ω, 1, 0),
[1 [2¬3] 1 \ 3] has level φ(ω, 1, 1),
[1 [2¬3] 1 \ 1 \ 2] has level φ(ω, 2, 0),
[1 [2¬3] 1 \ 1 \ 1 \ 2] has level φ(ω, 3, 0),
[1 [2¬3] 1 [2¬2] 2] has level φ(ω, ω, 0),
[1 [2¬3] 1 [1, 2 ¬2] 2] has level φ(ω, ω^ω, 0),
[1 [2¬3] 1 [1 [2¬3] 2 ¬2] 2] has level φ(ω, φ(ω, 0, 0), 0),
[1 [2¬3] 1 [1 [2¬3] 1 [1 [2¬3] 2 ¬2] 2 ¬2] 2] has level φ(ω, φ(ω, φ(ω, 0, 0), 0), 0).

[1 [2¬3] 1 [1¬3] 2] has level φ(ω+1, 0, 0),
[1 [2¬3] 1 [1¬3] 3] has level φ(ω+1, 0, 1),
[1 [2¬3] 1 [1¬3] 1 \ 2] has level φ(ω+1, 1, 0),
[1 [2¬3] 1 [1¬3] 1 [2¬2] 2] has level φ(ω+1, ω, 0),
[1 [2¬3] 1 [1¬3] 1 [1 [2¬3] 1 [1¬3] 2 ¬2] 2] has level φ(ω+1, φ(ω+1, 0, 0), 0),
[1 [2¬3] 1 [1¬3] 1 [1¬3] 2] has level φ(ω+2, 0, 0),
[1 [2¬3] 1 [1¬3] 1 [1¬3] 1 [1¬3] 2] has level φ(ω+3, 0, 0),
[1 [2¬3] 1 [2¬3] 2] has level φ(ω2, 0, 0),
[1 [2¬3] 1 [2¬3] 1 [2¬3] 2] has level φ(ω3, 0, 0),
[1 [3¬3] 2] has level φ(ω^2, 0, 0),
[1 [3¬3] 1 [1¬3] 2] has level φ(ω^2+1, 0, 0),
[1 [3¬3] 1 [2¬3] 2] has level φ(ω^2+ω, 0, 0),
[1 [3¬3] 1 [3¬3] 2] has level φ((ω^2)2, 0, 0),
[1 [4¬3] 2] has level φ(ω^3, 0, 0),
[1 [5¬3] 2] has level φ(ω^4, 0, 0),
[1 [1, 2 ¬3] 2] has level φ(ω^ω, 0, 0),
[1 [1 \ 2 ¬3] 2] has level φ(ε_0, 0, 0),
[1 [1 [1¬3] 2 ¬3] 2] has level φ(Γ_0, 0, 0),
[1 [1 [1 [1¬3] 2 ¬3] 2 ¬3] 2] has level φ(φ(Γ_0, 0, 0), 0, 0),
[1 [1 [1 [1 [1¬3] 2 ¬3] 2 ¬3] 2 ¬3] 2] has level φ(φ(φ(Γ_0, 0, 0), 0, 0), 0, 0).

The sequence of separators starting with the last three has limit ordinal φ(1, 0, 0, 0).

I am now into hyper-4 separators (hyperseparators of the form [X¬4], where X is an array or string of characters). I can now define the φ(1, 0, 0, 0) level separator:

{a, b [1 [1¬4] 2] 2} = {a ‹0 [1¬4] 2› b}
= {a ‹b ‹b ‹b ‹ ... ‹b ‹b¬3› b¬3› ... › b¬3› b¬3› b› b}

(with b+1 b’s from centre to right and b-1 ¬’s).

[1 [1¬4] 2] has level φ(1, 0, 0, 0),
[1 \ 2 [1¬4] 2] has level $\varepsilon_{\varphi(1, 0, 0, 0)+1}$ = φ(1, 0, 0, 0)^^ω,
[1 \ 1 \ 2 [1¬4] 2] has level $\zeta_{\varphi(1, 0, 0, 0)+1}$ = φ(2, φ(1, 0, 0, 0)+1),
[1 [2¬2] 2 [1¬4] 2] has level φ(ω, φ(1, 0, 0, 0)+1),
[1 [1 \ 2 ¬2] 2 [1¬4] 2] has level φ(ε_0, φ(1, 0, 0, 0)+1),
[1 [1 [1¬3] 2 ¬2] 2 [1¬4] 2] has level φ(Γ_0, φ(1, 0, 0, 0)+1),
[1 [1 [1¬4] 2 ¬2] 2 [1¬4] 2] has level φ(φ(1, 0, 0, 0), 1) = φ($\Gamma_{\varphi(1, 0, 0, 0)}$, 1)
(limit ordinal of φ(α, φ(1, 0, 0, 0)+1) = φ(α, φ(φ(1, 0, 0, 0), 0)+1) as α → φ(1, 0, 0, 0)),
[1 [1 [1 [1¬4] 2 ¬2] 2 [1¬4] 2 ¬2] 2 [1¬4] 2] has level φ(φ(φ(1, 0, 0, 0), 1), 0),
[1 [1 [1 [1 [1¬4] 2 ¬2] 2 [1¬4] 2 ¬2] 2 [1¬4] 2 ¬2] 2 [1¬4] 2] has level
φ(φ(φ(φ(1, 0, 0, 0), 1), 0), 0),
[1 [1¬3] 2 [1¬4] 2] has level $\Gamma_{\varphi(1, 0, 0, 0)+1}$ = φ(1, 0, φ(1, 0, 0, 0)+1),
[1 [1¬3] 1 \ 2 [1¬4] 2] has level φ(1, 1, φ(1, 0, 0, 0)+1),
[1 [1¬3] 1 [1¬3] 2 [1¬4] 2] has level φ(2, 0, φ(1, 0, 0, 0)+1),
[1 [2¬3] 2 [1¬4] 2] has level φ(ω, 0, φ(1, 0, 0, 0)+1),
[1 [1 \ 2 ¬3] 2 [1¬4] 2] has level φ(ε_0, 0, φ(1, 0, 0, 0)+1),
[1 [1 [1¬3] 2 ¬3] 2 [1¬4] 2] has level φ(Γ_0, 0, φ(1, 0, 0, 0)+1),
[1 [1 [1¬4] 2 ¬3] 2 [1¬4] 2] has level φ(φ(1, 0, 0, 0), 0, 1)
(limit ordinal of φ(α, 0, φ(1, 0, 0, 0)+1) = φ(α, 0, φ(φ(1, 0, 0, 0), 0, 0)+1)
as α → φ(1, 0, 0, 0)),
[1 [1 [1 [1¬4] 2 ¬3] 2 [1¬4] 2 ¬3] 2 [1¬4] 2] has level φ(φ(φ(1, 0, 0, 0), 0, 1), 0, 0),
[1 [1 [1 [1 [1¬4] 2 ¬3] 2 [1¬4] 2 ¬3] 2 [1¬4] 2 ¬3] 2 [1¬4] 2] has level
φ(φ(φ(φ(1, 0, 0, 0), 0, 1), 0, 0), 0, 0).

[1 [1¬4] 3] has level φ(1, 0, 0, 1),
[1 [1¬4] 4] has level φ(1, 0, 0, 2),
[1 [1¬4] 1 [1 \ 2] 2] has level φ(1, 0, 0, ε_0),
[1 [1¬4] 1 [1 [1¬4] 2] 2] has level φ(1, 0, 0, φ(1, 0, 0, 0)),
[1 [1¬4] 1 \ 2] has level φ(1, 0, 1, 0),
[1 [1¬4] 1 \ 3] has level φ(1, 0, 1, 1),
[1 [1¬4] 1 \ 1 \ 2] has level φ(1, 0, 2, 0),
[1 [1¬4] 1 \ 1 \ 1 \ 2] has level φ(1, 0, 3, 0),
[1 [1¬4] 1 [2¬2] 2] has level φ(1, 0, ω, 0),
[1 [1¬4] 1 [1 \ 2 ¬2] 2] has level φ(1, 0, ε_0, 0),
[1 [1¬4] 1 [1 [1¬4] 2 ¬2] 2] has level φ(1, 0, φ(1, 0, 0, 0), 0),
[1 [1¬4] 1 [1¬3] 2] has level φ(1, 1, 0, 0),
[1 [1¬4] 1 [1¬3] 3] has level φ(1, 1, 0, 1),
[1 [1¬4] 1 [1¬3] 1 \ 2] has level φ(1, 1, 1, 0),
[1 [1¬4] 1 [1¬3] 1 [1¬3] 2] has level φ(1, 2, 0, 0),
[1 [1¬4] 1 [1¬3] 1 [1¬3] 1 [1¬3] 2] has level φ(1, 3, 0, 0),
[1 [1¬4] 1 [2¬3] 2] has level φ(1, ω, 0, 0),
[1 [1¬4] 1 [1 \ 2 ¬3] 2] has level φ(1, ε_0, 0, 0),
[1 [1¬4] 1 [1 [1¬4] 2 ¬3] 2] has level φ(1, φ(1, 0, 0, 0), 0, 0),
[1 [1¬4] 1 [1 [1¬4] 1 [1 [1¬4] 2 ¬3] 2 ¬3] 2] has level φ(1, φ(1, φ(1, 0, 0, 0), 0, 0), 0, 0).

[1 [1¬4] 1 [1¬4] 2] has level φ(2, 0, 0, 0),
[1 [1¬4] 1 [1¬4] 3] has level φ(2, 0, 0, 1),

[1 [1¬4] 1 [1¬4] 1 \ 2] has level φ(2, 0, 1, 0),
[1 [1¬4] 1 [1¬4] 1 [1¬3] 2] has level φ(2, 1, 0, 0),
[1 [1¬4] 1 [1¬4] 1 [1¬4] 2] has level φ(3, 0, 0, 0),
[1 [1¬4] 1 [1¬4] 1 [1¬4] 1 [1¬4] 2] has level φ(4, 0, 0, 0),
[1 [1¬4] 1 [1¬4] ... [1¬4] 1 [1¬4] 2] (with n [1¬4]'s) has level φ(n, 0, 0, 0),
[1 [1¬4] 1 [1¬4] ... [1¬4] 1 [1¬3] 1 [1¬3] ... [1¬3] 1 \ 1 \ ... \ 1 \ d] (with a [1¬4]'s, b [1¬3]'s, c \'s)
has level φ(a, b, c, d-2).

[1 [2¬4] 2] has level φ(ω, 0, 0, 0),
[1 [2¬4] 1 [1¬4] 2] has level φ(ω+1, 0, 0, 0),
[1 [2¬4] 1 [2¬4] 2] has level φ(ω2, 0, 0, 0),
[1 [3¬4] 2] has level φ(ω^2, 0, 0, 0),
[1 [4¬4] 2] has level φ(ω^3, 0, 0, 0),
[1 [1, 2 ¬4] 2] has level φ(ω^ω, 0, 0, 0),
[1 [1 \ 2 ¬4] 2] has level φ(ε_0, 0, 0, 0),
[1 [1 [1¬3] 2 ¬4] 2] has level φ(Γ_0, 0, 0, 0),
[1 [1 [1¬4] 2 ¬4] 2] has level φ(φ(1, 0, 0, 0), 0, 0, 0),
[1 [1 [1 [1¬4] 2 ¬4] 2 ¬4] 2] has level φ(φ(φ(1, 0, 0, 0), 0, 0, 0), 0, 0, 0),
[1 [1 [1 [1 [1¬4] 2 ¬4] 2 ¬4] 2 ¬4] 2] has level φ(φ(φ(φ(1, 0, 0, 0), 0, 0, 0), 0, 0, 0), 0, 0, 0).
The sequence of separators starting with the last three has limit ordinal φ(1, 0, 0, 0, 0).

The φ(1, 0, 0, 0, 0) level separator is defined as follows:
{a, b [1 [1¬5] 2] 2} = {a ‹0 [1¬5] 2› b}
= {a ‹b ‹b ‹b ‹ ... ‹b ‹b¬4› b¬4› ... › b¬4› b¬4› b› b}
(with b+1 b's from centre to right and b-1 ¬'s).

Each increment of the number to the right of the ¬ symbol adds one to the number of parameters used in the Veblen function.
[1 [1¬2] 2] which 'drops down to' [1 \ 2] has level θ(1) = φ(1, 0) = ε_0,
[1 [1¬3] 2] has level θ(Ω) = φ(1, 0, 0) = Γ_0,
[1 [1¬4] 2] has level θ(Ω^2) = φ(1, 0, 0, 0),
[1 [1¬5] 2] has level θ(Ω^3) = φ(1, 0, 0, 0, 0),
[1 [1¬6] 2] has level θ(Ω^4) = φ(1, 0, 0, 0, 0, 0),
[1 [1¬7] 2] has level θ(Ω^5) = φ(1, 0, 0, 0, 0, 0, 0).

In general,
[1 [1¬n] 2] has level θ(Ω^(n-2)) = φ(1, 0, 0, ... , 0) (with n-1 zeroes).

With n_2 \'s, n_3 [1¬3]'s, n_4 [1¬4]'s, ... , n_k [1¬k]'s,
[1 [1¬k] 1 [1¬k] ... [1¬k] 1 ...... 1 [1¬3] 1 [1¬3] ... [1¬3] 1 \ 1 \ ... \ 1 \ n_1]
has level φ(n_k, ... , n_3, n_2, n_1-2).
Replacing each [1¬b] by [2¬b] for some 1 ≤ b ≤ k, multiplies the n_b in the Veblen function by ω, and changing each [1¬b] by [a¬b], multiplies the n_b by ω^(a-1).

Where 0, ... , 0 represents n-1 zeroes,
[1 [1¬n] 2] has level φ(1, 0, ... , 0),
[1 [1¬n] 1 [1¬n] ... [1¬n] 1 [1¬n] 2] (with m [1¬n]'s) has level φ(m, 0, ... , 0),
[1 [2¬n] 2] has level φ(ω, 0, ... , 0),
[1 [1, 2 ¬n] 2] has level φ(ω^ω, 0, ... , 0),

[1 [1 \ 2 ¬n] 2] has level φ(ε_0, 0, ... , 0),
[1 [1 [1¬3] 2 ¬n] 2] has level φ(Γ_0, 0, ... , 0),
[1 [1 [1¬4] 2 ¬n] 2] has level φ(φ(1, 0, 0, 0), 0, ... , 0),
[1 [1 [1¬n] 2 ¬n] 2] has level φ(φ(1, 0, ... , 0), 0, ... , 0),
[1 [1 [1 [1¬n] 2 ¬n] 2 ¬n] 2] has level φ(φ(φ(1, 0, ... , 0), 0, ... , 0), 0, ... , 0),
[1 [1 [1 [1 [1¬n] 2 ¬n] 2 ¬n] 2 ¬n] 2] has level φ(φ(φ(φ(1,0,...,0), 0,...,0), 0,...,0), 0,...,0).

The sequence of separators starting with the last three has limit ordinal φ(1, 0, 0, ... , 0) (with n zeroes).

In general, the φ(1, 0, ... , 0) (with n-1 zeroes, n ≥ 3) level separator is defined as follows:

{a, b [1 [1¬n] 2] 2} = {a ‹0 [1¬n] 2› b}
= {a ‹b ‹b ‹b ‹ ... ‹b ‹b¬n-1› b¬n-1› ... › b¬n-1› b¬n-1› b› b}
(with b+1 b's from centre to right and b-1 ¬'s).

This bears a similarity to

{a, b [1 \ n] 2} = {a ‹0 \ n› b}
= {a ‹b ‹b ‹ ... ‹b ‹b \ n-1› b \ n-1› ... › b \ n-1› b \ n-1› b}
(with b b's from centre to right and b-1 \'s).

The level of recursion of the function

f(n) = {3, n [1 [1¬n] 2] 2}

would have ordinal

θ(Ω^ω) = φ(1, 0, 0, ... , 0)   (with ω zeroes),

which is the small Veblen ordinal.

The backslash and hyperseparators (those containing a ¬ symbol) of the form [1¬n] are known as nesting separators. When an angle bracket array (between the numbers a and b) contains a string of 1's (after an initial 0, or a single 0) and there is a nesting separator found immediately prior to the next non-1 entry, the 1 (or 0 if it is the first entry) to the left of the nesting separator gets replaced by a angle bracket array that is nested to b layers with 0 on top (b-1 layers if the nesting separator is a backslash).

The additional Angle Bracket (B) Rules are much modified but remain at three rules:-

Rule B1 (first entry is 0 or 1, followed by negation (¬) symbol):

'a ‹0¬n› b' = 'a',
'a ‹1¬n› b' = 'a [1¬n] a [1¬n] ... [1¬n] a'   (with b a's),
'a ‹1¬2› b' = 'a \ a \ ... \ a'   (with b a's).

Rule B2 (Rule B1 does not apply, first entry is 0, separator immediately prior to next non-1 entry (c) is [1¬2] or backslash):

'a ‹ 0 [A_1] 1 [A_2] 1 ... [A_p] 1 \ c # ¬d› b' = 'a ‹R_b ¬d› b',

where R_n = 'b ‹A_1'› b [A_1] b ‹A_2'› b [A_2] ... b ‹A_p'› b [A_p] b ‹R_{n-1}› b \ c-1 #'   (n > 1),
R_1 = '0'.

(p is a non-negative integer. '¬d' is removed in the case d = 1.)

If p = 0 then

'a ‹0 \ c # ¬d› b' = 'a ‹R_b ¬d› b',

where R_n = 'b ‹R_{n-1}› b \ c-1 #'   (n > 1),
R_1 = '0'.

Rule B3 (Rules B1-2 do not apply, first entry is 0, separator immediately prior to next non-1 entry (c) is [1¬k], where k ≥ 3 (i.e. higher than backslash)):

‘a ‹ 0 [A_1] 1 [A_2] 1 ... [A_p] 1 [1¬k] c # ¬d› b’ = ‘a ‹R ¬d› b’,

where R = ‘b ‹A_1’› b [A_1] b ‹A_2’› b [A_2] ... b ‹A_p’› b [A_p] b ‹R_b› b [1¬k] c-1 #’,

R_n = ‘b ‹A_1’› b [A_1] b ‹A_2’› b [A_2] ... b ‹A_p’› b [A_p] b ‹R_{n-1}› b [1¬k] c-1 # ¬ k-1’ (n > 1),

R_1 = ‘0’.

(p is a non-negative integer. ‘¬d’ is removed in the case d = 1.)

If p = 0 then

‘a ‹0 [1¬k] c # ¬d› b’ = ‘a ‹R ¬d› b’,

where R = ‘b ‹R_b› b [1¬k] c-1 #’,

R_n = ‘b ‹R_{n-1}› b [1¬k] c-1 # ¬ k-1’ (n > 1),

R_1 = ‘0’.

Notes:

1. A_1, A_2, ... , A_p are strings of characters within separators of [1¬k] level or above (backslash if Rule B2 applies).
2. A_1’, A_2’, ... , A_p’ are strings of characters within angle brackets that are identical to the strings A_1, A_2, ... , A_p respectively except that the first entries of each have been reduced by 1. If A_i (for some 1 ≤ i ≤ p) begins with 1, then A_i’ begins with 0.
3. R_n is an iterating string building function which creates and nests the same string of characters around itself n-1 times before being replaced by the string ‘0’.
4. # is a string of characters representing the remainder of the array (if it exists), which excludes the ¬ symbol (though it can include the ¬ symbol followed by a number in the regular Angle Bracket Rules A4-5).
5. The backslash (\) is a hyperseparator. The ¬ symbol enclosed by one pair of square brackets (e.g. [A¬n], where A is a string of characters) is also a hyperseparator. The ¬ symbol itself is an even more special symbol. All other separators are normal separators.
6. The comma is used as shorthand for the [1] separator.
7. The backslash (\) is used as shorthand for the [1¬2] separator.

The regular Angle Bracket Rule A2 is modified as follows:

‘a ‹# [A] 1› b’ = ‘a ‹#› b’,

‘a ‹# [A] 1 ¬ n› b’ = ‘a ‹# ¬ n› b’,

‘a ‹# ¬ 1› b’ = ‘a ‹#› b’.

When [A] is a normal separator and [B] is a hyperseparator, or level of [A] is less than level of [B],

‘a ‹# [A] 1 [B] #*› b’ = ‘a ‹# [B] #*› b’,

‘a ‹# [A] 1 [B] #* ¬ n› b’ = ‘a ‹# [B] #* ¬ n› b’.

Remove trailing 1’s.

This is the second stage of my Hyper-Nested Array Notation. The limit ordinal of this notation is θ(Ω^ω), the small Veblen ordinal.

The levels of two hyperseparators [A] = [X¬m] and [B] = [Y¬n] are determined by the values of m and n; if m < n then [A] ranks lower than [B] and the ‘[A] 1’ string is deleted. If m = n then the levels of [A] and [B] are determined by the ordinal levels of the two normal separators [X] and [Y] – if the level of [X] is less than that of [Y], then [A] ranks lower than [B].

Yet, I can take this further by turning the number to the right of the ¬ into an array!

{a, b [1 [1¬1,2] 2] 2} = {a ‹0 [1¬1,2] 2› b}

= {a ‹b ‹b¬b› b› b}.

It is a θ(Ω^ω)-recursive function.

Notice the similarity to

{a, b [1 \ 1, 2] 2} = {a ‹0 \ 1, 2› b}

= {a ‹b \ b› b}.

Higher separators involving ¬ are introduced based on these similarities with \.

While the backslash and separators of the form [X¬n] or [X¬Y] (when Y is an array) are hyperseparators, the ¬ symbol (always enclosed by a minimum of two square or angle brackets) takes things to a whole new level, as it is a 2-hyperseparator (hyper-hyperseparator, not to be confused with a hyper-2 separator.) Nesting separators include those of the form [1 ¬ n #] (where n ≥ 2 and # is rest of array).

The number immediately to the left of the ¬ symbol in the string replacement equation

‘a ‹0 [1¬1,2] 2› b’ = ‘a ‹b ‹b¬b› b› b’

becomes ‘b’, since when [A] is the ¬ sign, the string A’ is set to ‘0’, just as with the comma and backslash symbols (this makes ‘b ‹0› b’ = ‘b’). While the comma and backslash symbols are the first separator and hyperseparator respectively, the ¬ symbol is the first 2-hyperseparator. The backslash is written as shorthand for [1¬2] in the notation; when [A] is a backslash, ‘b ‹A’› b’ = ‘b ‹0¬2› b’ = ‘b’.

The Complex Projective 4-Space webpage “TREE(3) and impartial games” (http://cp4space.wordpress.com/2012/12/19/fast-growing-2/) states that Friedman’s TREE sequence (finite form of Kruskal’s Tree Theorem) grows so astoundingly rapidly that the first three values are:

TREE(1) = 1,

TREE(2) = 3,

TREE(3) > tree^(tree^(tree^(tree^(tree^8(7)) (7)) (7)) (7)) (7),

where the slightly slower growing tree function (lower case) grows at least as quickly as

f(n) = {3, n [1 [1¬1,2] 2] 2}

which is at the level of the small Veblen ordinal (much greater than Γ_0). Royce Peng has investigated that the growth rate of the slower tree function is at the θ(Ω^(n-2)) level for n ≥ 4, meaning that

tree(n) > {3, 3 [1 [1 ¬ n] 2] 2} (for n ≥ 4).

From the information above, I find that

tree(7) > {3, 6 [1 [1¬1,2] 2] 2},

tree^2(7) = tree(tree(7))

> {3, {3, 6 [1 [1¬1,2] 2] 2} [1 [1¬1,2] 2] 2}

> {3, 3, 2 [1 [1¬1,2] 2] 2},

tree^8(7) > {3, 9, 2 [1 [1¬1,2] 2] 2},

tree^(tree^8(7)) (7) > {3, {3, 9, 2 [1 [1¬1,2] 2] 2}, 2 [1 [1¬1,2] 2] 2}

> {3, 3, 3 [1 [1¬1,2] 2] 2},

tree^(tree^(tree^8(7)) (7)) (7) > {3, 4, 3 [1 [1¬1,2] 2] 2},

which means that

TREE(3) > {3, 6, 3 [1 [1¬1,2] 2] 2}.

Even I have great difficulty in attempting to grasp the sheer magnitude of this number, to say nothing of TREE(4) and higher terms of the TREE sequence.

Since we have reached the limit of the Veblen function it is probably better to use a different notation to describe ordinals higher than the small Veblen ordinal, for example, an ordinal collapsing function (θ) which uses Ω to denote the first uncountable ordinal. It relates to the Veblen function as follows:

$\theta(0) = 1$,
$\theta(1) = \varphi(1, 0) = \varepsilon_0$,
$\theta(2) = \varphi(2, 0) = \zeta_0$,
$\theta(3) = \varphi(3, 0)$,
$\theta(\alpha) = \varphi(\alpha, 0)$ (α < Ω),
$\theta(\omega) = \varphi(\omega, 0)$,
$\theta(\Omega) = \theta(\theta(\theta(...\theta(0)...)))$ (with ω θ's)
$= \varphi(1, 0, 0) = \Gamma_0$,
$\theta(\Omega2) = \theta(\Omega+\theta(\Omega+\theta(\Omega+...\theta(\Omega)...)))$ (with ω θ's)
$= \varphi(2, 0, 0)$,
$\theta(\Omega3) = \theta(\Omega2+\theta(\Omega2+\theta(\Omega2+...\theta(\Omega2)...)))$ (with ω θ's)
$= \varphi(3, 0, 0)$,
$\theta(\Omega\alpha) = \varphi(\alpha, 0, 0)$ (α < Ω),
$\theta(\Omega^2) = \theta(\Omega\theta(\Omega\theta(\Omega...\theta(\Omega)...)))$ (with ω θ's)
$= \varphi(1, 0, 0, 0)$,
$\theta(\Omega^3) = \theta((\Omega^2)\theta((\Omega^2)\theta((\Omega^2)...\theta(\Omega^2)...)))$ (with ω θ's)
$= \varphi(1, 0, 0, 0, 0)$,
$\theta(\Omega^n) = \varphi(1, 0, 0, ... , 0)$ (with n+1 zeroes),
$\theta(\Omega^\omega) = \varphi(1, 0, 0, ... , 0)$ (with ω zeroes, small Veblen ordinal),
$\theta(\Omega^\Omega) = \theta(\Omega^{\theta(\Omega^{\theta(\Omega^{...\theta(\Omega)...})})})$ (with ω θ's, large Veblen ordinal).

The theta function can have two arguments in order to express the intermediate ordinals, for example,

$\theta(0, \alpha) = \omega^\alpha$,
$\theta(1, \alpha) = \varphi(1, \alpha) = \varepsilon_\alpha$,
$\theta(2, \alpha) = \varphi(2, \alpha) = \zeta_\alpha$,
$\theta(3, \alpha) = \varphi(3, \alpha)$,
$\theta(\alpha, \beta) = \varphi(\alpha, \beta)$,
$\theta(\Omega, \alpha) = \Gamma_\alpha$,

where α and β are in terms of the finite numbers, ω, addition, multiplication, exponentiation and the θ function itself (Ω is only used in the first argument of the θ function). This enables us to dispense with the Veblen function and the epsilon numbers and express ordinals using finite numbers, ω, addition, multiplication, exponentiation and θ itself. When the second argument of θ is zero, it can be omitted:

$\theta(\alpha, 0) = \theta(\alpha)$ (for all α, including α ≥ Ω).

In general, the two-argument θ function

$\theta((\Omega^n)\alpha_n + ... + (\Omega^2)\alpha_2 + \Omega\alpha_1 + \alpha_0, \beta) = \varphi(\alpha_n, ... , \alpha_2, \alpha_1, \alpha_0, \beta)$.

Each of the following separators can be expressed using the single-argument theta function, since the second argument of the two-argument function would be 0 in every case. If the 2's at the end of each separator were replaced by n's (with n > 2), the second arguments would be n-2. If the 2's at the end were replaced by an array X where [X] has level α ≥ ω, the second arguments would be α.

[1 \ 2] has level $\theta(1) = \varepsilon_0$,
[1 \ 1 \ 2] has level θ(2),
[1 \ 1 \ 1 \ 2] has level θ(3),
[1 [2¬2] 2] has level θ(ω),

[1 [2¬2] 1 \ 2] has level θ(ω+1),
[1 [2¬2] 1 \ 1 \ 2] has level θ(ω+2),
[1 [2¬2] 1 [2¬2] 2] has level θ(ω2),
[1 [2¬2] 1 [2¬2] 1 [2¬2] 2] has level θ(ω3),
[1 [3¬2] 2] has level θ(ω^2),
[1 [4¬2] 2] has level θ(ω^3),
[1 [1, 2 ¬2] 2] has level θ(ω^ω),
[1 [1 [2] 2 ¬2] 2] has level θ(ω^ω^ω),
[1 [1 [1, 2] 2 ¬2] 2] has level θ(ω^ω^ω^ω),
[1 [1 \ 2 ¬2] 2] has level θ(θ(1)),
[1 [1 [1 \ 2 ¬2] 2 ¬2] 2] has level θ(θ(θ(1))),
[1 [1 [1 [1 \ 2 ¬2] 2 ¬2] 2 ¬2] 2] has level θ(θ(θ(θ(1)))).

[1 [1¬3] 2] has level θ(Ω) = Γ_0,
[1 [1¬3] 1 \ 2] has level θ(Ω+1),
[1 [1¬3] 1 \ 1 \ 2] has level θ(Ω+2),
[1 [1¬3] 1 [2¬2] 2] has level θ(Ω+ω),
[1 [1¬3] 1 [3¬2] 2] has level θ(Ω+ω^2),
[1 [1¬3] 1 [1, 2 ¬2] 2] has level θ(Ω+ω^ω),
[1 [1¬3] 1 [1 [1¬3] 2 ¬2] 2] has level θ(Ω+θ(Ω)),
[1 [1¬3] 1 [1 [1¬3] 1 [1 [1¬3] 2 ¬2] 2 ¬2] 2] has level θ(Ω+θ(Ω+θ(Ω))),
[1 [1¬3] 1 [1¬3] 2] has level θ(Ω2),
[1 [1¬3] 1 [1¬3] 1 [1 [1¬3] 1 [1¬3] 2 ¬2] 2] has level θ(Ω2+θ(Ω2)),
[1 [1¬3] 1 [1¬3] 1 [1¬3] 2] has level θ(Ω3),
[1 [2¬3] 2] has level θ(Ωω),
[1 [2¬3] 1 [1¬3] 2] has level θ(Ω(ω+1)),
[1 [2¬3] 1 [2¬3] 2] has level θ(Ωω2),
[1 [2¬3] 1 [2¬3] 1 [2¬3] 2] has level θ(Ωω3),
[1 [3¬3] 2] has level θ(Ωω^2),
[1 [4¬3] 2] has level θ(Ωω^3),
[1 [1, 2 ¬3] 2] has level θ(Ωω^ω),
[1 [1 \ 2 ¬3] 2] has level θ(Ωθ(1)),
[1 [1 [1¬3] 2 ¬3] 2] has level θ(Ωθ(Ω)),
[1 [1 [1 [1¬3] 2 ¬3] 2 ¬3] 2] has level θ(Ωθ(Ωθ(Ω))).

[1 [1¬4] 2] has level θ(Ω^2),
[1 [1¬4] 1 \ 2] has level θ(Ω^2+1),
[1 [1¬4] 1 [2¬2] 2] has level θ(Ω^2+ω),
[1 [1¬4] 1 [1 [1¬4] 2 ¬2] 2] has level θ(Ω^2+θ(Ω^2)),
[1 [1¬4] 1 [1 [1¬4] 1 [1 [1¬4] 2 ¬2] 2 ¬2] 2] has level θ(Ω^2+θ(Ω^2+θ(Ω^2))),
[1 [1¬4] 1 [1¬3] 2] has level θ(Ω^2+Ω),
[1 [1¬4] 1 [2¬3] 2] has level θ(Ω^2+Ωω),
[1 [1¬4] 1 [1 [1¬4] 2 ¬3] 2] has level θ(Ω^2+Ωθ(Ω^2)),
[1 [1¬4] 1 [1 [1¬4] 1 [1 [1¬4] 2 ¬3] 2 ¬3] 2] has level θ(Ω^2+Ωθ(Ω^2+Ωθ(Ω^2))),
[1 [1¬4] 1 [1¬4] 2] has level θ((Ω^2)2),
[1 [1¬4] 1 [1¬4] 1 [1¬4] 2] has level θ((Ω^2)3),
[1 [2¬4] 2] has level θ((Ω^2)ω),
[1 [3¬4] 2] has level θ((Ω^2)ω^2),
[1 [1, 2 ¬4] 2] has level θ((Ω^2)ω^ω),

[1 [1 \ 2 ¬4] 2] has level θ((Ω^2)θ(1)),
[1 [1 [1¬3] 2 ¬4] 2] has level θ((Ω^2)θ(Ω)),
[1 [1 [1¬4] 2 ¬4] 2] has level θ((Ω^2)θ(Ω^2)),
[1 [1 [1 [1¬4] 2 ¬4] 2 ¬4] 2] has level θ((Ω^2)θ((Ω^2)θ(Ω^2))),
[1 [1 [1 [1 [1¬4] 2 ¬4] 2 ¬4] 2 ¬4] 2] has level θ((Ω^2)θ((Ω^2)θ((Ω^2)θ(Ω^2)))).

[1 [1¬5] 2] has level θ(Ω^3),
[1 [1¬6] 2] has level θ(Ω^4),
[1 [1¬7] 2] has level θ(Ω^5).

Each \ (after a 1) added to the separator immediately before the final 2 and closed square bracket adds 1 to the number inside the θ function. Each [2¬2] similarly adds ω, each [3¬2] adds ω^2, each [1,2¬2] adds ω^ω, each [1¬3] adds Ω, each [2¬3] adds Ωω, each [1¬4] adds Ω^2, etc. If [1 [X] 2] and [1 [Y] 2] have levels of θ(α) and θ(β) respectively (where α ≥ β), then [1 [X] 1 [Y] 2] would have level θ(α+β). We can spot a pattern which enables us to deduce what the ordinal levels are for the higher separators beyond the small Veblen ordinal.

[1 [1¬1,2] 2] has level θ(Ω^ω),
[1 [1¬1,2] 1 \ 2] has level θ(Ω^ω+1),
[1 [1¬1,2] 1 [2¬2] 2] has level θ(Ω^ω+ω),
[1 [1¬1,2] 1 [1¬3] 2] has level θ(Ω^ω+Ω),
[1 [1¬1,2] 1 [2¬3] 2] has level θ(Ω^ω+Ωω),
[1 [1¬1,2] 1 [1¬4] 2] has level θ(Ω^ω+Ω^2),
[1 [1¬1,2] 1 [1¬5] 2] has level θ(Ω^ω+Ω^3),
[1 [1¬1,2] 1 [1¬1,2] 2] has level θ((Ω^ω)2),
[1 [1¬1,2] 1 [1¬1,2] 1 [1¬1,2] 2] has level θ((Ω^ω)3),
[1 [2¬1,2] 2] has level θ((Ω^ω)ω),
[1 [1 \ 2 ¬1,2] 2] has level θ((Ω^ω)θ(1)),
[1 [1 [1¬3] 2 ¬1,2] 2] has level θ((Ω^ω)θ(Ω)),
[1 [1 [1¬1,2] 2 ¬1,2] 2] has level θ((Ω^ω)θ(Ω^ω)),
[1 [1 [1 [1¬1,2] 2 ¬1,2] 2 ¬1,2] 2] has level θ((Ω^ω)θ((Ω^ω)θ(Ω^ω))).

[1 [1¬2,2] 2] has level θ(Ω^(ω+1)),
[1 [1¬2,2] 1 \ 2] has level θ(Ω^(ω+1)+1),
[1 [1¬2,2] 1 [1¬2,2] 2] has level θ((Ω^(ω+1))2),
[1 [2¬2,2] 2] has level θ((Ω^(ω+1))ω),
[1 [1 [1¬2,2] 2 ¬2,2] 2] has level θ((Ω^(ω+1))θ(Ω^(ω+1))),
[1 [1 [1 [1¬2,2] 2 ¬2,2] 2 ¬2,2] 2] has level θ((Ω^(ω+1))θ((Ω^(ω+1))θ(Ω^(ω+1)))).

[1 [1 ¬ 3, 2] 2] has level θ(Ω^(ω+2)),
[1 [1 ¬ 4, 2] 2] has level θ(Ω^(ω+3)),
[1 [1 ¬ 1, 3] 2] has level θ(Ω^(ω2)),
[1 [1 ¬ 2, 3] 2] has level θ(Ω^(ω2+1)),
[1 [1 ¬ 1, 4] 2] has level θ(Ω^(ω3)),
[1 [1 ¬ 1, 1, 2] 2] has level θ(Ω^ω^2),
[1 [1 ¬ 1, 1, 3] 2] has level θ(Ω^((ω^2)2)),
[1 [1 ¬ 1, 1, 1, 2] 2] has level θ(Ω^ω^3),
[1 [1 ¬ 1 [2] 2] 2] has level θ(Ω^ω^ω),
[1 [1 ¬ 1 [3] 2] 2] has level θ(Ω^ω^ω^2),

[1 [1 ¬ 1 [1, 2] 2] 2] has level θ(Ω^ω^ω^ω),
[1 [1 ¬ 1 [1 [2] 2] 2] 2] has level θ(Ω^ω^ω^ω^ω),
[1 [1 ¬ 1 [1 [1, 2] 2] 2] 2] has level θ(Ω^ω^ω^ω^ω^ω).

[1 [1 ¬ 1 \ 2] 2] has level θ(Ω^θ(1)) = θ(Ω^ε_0),
[1 [1 ¬ 1 \ 1 \ 2] 2] has level θ(Ω^θ(2)),
[1 [1 ¬ 1 [2¬2] 2] 2] has level θ(Ω^θ(ω)),
[1 [1 ¬ 1 [1, 2 ¬2] 2] 2] has level θ(Ω^θ(ω^ω)),
[1 [1 ¬ 1 [1 \ 2 ¬2] 2] 2] has level θ(Ω^θ(θ(1))),
[1 [1 ¬ 1 [1 ¬ 3] 2] 2] has level θ(Ω^θ(Ω)) = θ(Ω^Γ_0),
[1 [1 ¬ 1 [1 ¬ 4] 2] 2] has level θ(Ω^θ(Ω^2)),
[1 [1 ¬ 1 [1 ¬ 5] 2] 2] has level θ(Ω^θ(Ω^3)),
[1 [1 ¬ 1 [1 ¬ 1, 2] 2] 2] has level θ(Ω^θ(Ω^ω)),
[1 [1 ¬ 1 [1 ¬ 2, 2] 2] 2] has level θ(Ω^θ(Ω^(ω+1))),
[1 [1 ¬ 1 [1 ¬ 1, 3] 2] 2] has level θ(Ω^θ(Ω^(ω2))),
[1 [1 ¬ 1 [1 ¬ 1, 1, 2] 2] 2] has level θ(Ω^θ(Ω^ω^2)),
[1 [1 ¬ 1 [1 ¬ 1 [2] 2] 2] 2] has level θ(Ω^θ(Ω^ω^ω)).

[1 [1 ¬ 1 [1 ¬ 1 \ 2] 2] 2] has level θ(Ω^θ(Ω^θ(1))),
[1 [1 ¬ 1 [1 ¬ 1 [1 ¬ 3] 2] 2] 2] has level θ(Ω^θ(Ω^θ(Ω))),
[1 [1 ¬ 1 [1 ¬ 1 [1 ¬ 1, 2] 2] 2] 2] has level θ(Ω^θ(Ω^θ(Ω^ω))),
[1 [1 ¬ 1 [1 ¬ 1 [1 ¬ 1 \ 2] 2] 2] 2] has level θ(Ω^θ(Ω^θ(Ω^θ(1)))),
[1 [1 ¬ 1 [1 ¬ 1 [1 ¬ 1 [1 ¬ 1 \ 2] 2] 2] 2] 2] has level θ(Ω^θ(Ω^θ(Ω^θ(Ω^θ(1))))),
[1 [1 ¬ 1 [1 ¬ 1 [1 ¬ 1 [1 ¬ 1 [1 ¬ 1 \ 2] 2] 2] 2] 2] 2] has level θ(Ω^θ(Ω^θ(Ω^θ(Ω^θ(Ω^θ(1)))))).

The sequence of separators starting with the last three has limit ordinal θ(Ω^Ω), which is the large Veblen ordinal.

Hyperseparators of the form [1 ¬ 1 [X] n #] (where X is an array, n ≥ 2 and # is remainder of array) are known as branching separators, since the execution of the Angle Bracket Rules ‘branches’ to the right of the ¬ symbol. Branching separators and nesting separators make up the two types of special separators that feature at this level of the Hyper-Nested Array Notation; special separators are either backslashes or hyperseparators of the form [1¬Y] where Y is an array.

When [X] is not a special separator and # is rest of array,

{a, b [1 [1 ¬ 1 [X] c #] 2] 2} = {a ‹0 [1 ¬ 1 [X] c #] 2› b}
= {a ‹b ‹ b ¬ b ‹X’› b [X] c-1 # › b› b},

where X’ is identical to X except that the first entry has been reduced by 1.

Set c = 2 and # blank. The above becomes

{a, b [1 [1 ¬ 1 [X] 2] 2] 2} = {a ‹0 [1 ¬ 1 [X] 2] 2› b}
= {a ‹b ‹b ¬ b ‹X’› b› b› b}.

When [X] is a special separator placed prior to the next non-1 entry (after the initial entry of 0), special rules apply to the right of the ¬ symbol (we use a version of Angle Bracket Rules B2-3 rather than Rule A4). For the case [X] = [1¬2] which ‘drops down’ to a backslash (\), the θ(Ω^θ(1)) level separator is defined as follows:

{a, b [1 [1 ¬ 1 \ 2] 2] 2} = {a ‹0 [1 ¬ 1 \ 2] 2› b}
= {a ‹b ‹b ¬ R_b› b› b},

where R_n = ‘b ‹R_{n-1}› b’,

R_1 = ‘0’.

Only one iterating string building function (R_n) is necessary and I have tried to simplify the rules as much as possible.

The θ(Ω^θ(2)) level separator is defined as follows:

{a, b [1 [1 ¬ 1 \ 1 \ 2] 2] 2} = {a ‹0 [1 ¬ 1 \ 1 \ 2] 2› b}

= {a ‹b ‹b ¬ R_b› b› b},

where R_n = ‘b \ b ‹R_{n-1}› b’,

R_1 = ‘0’.

For the case [X] = [1¬k] (where k ≥ 3), the θ(Ω^θ(Ω^(k-2))) level separator is defined as follows:

{a, b [1 [1 ¬ 1 [1¬k] 2] 2] 2} = {a ‹0 [1 ¬ 1 [1¬k] 2] 2› b}

= {a ‹b ‹ b ¬ b ‹R_b› b › b› b},

where R_n = ‘b ‹R_{n-1}› b ¬ k-1’,

R_1 = ‘0’.

For the case [X] = [1 ¬ 1 [Y] 2], where [Y] is not a special separator (e.g. of the form [1¬Z] where Z is an array),

{a, b [1 [1 ¬ 1 [1 ¬ 1 [Y] 2] 2] 2] 2} = {a ‹0 [1 ¬ 1 [1 ¬ 1 [Y] 2] 2] 2› b}

= {a ‹b ‹ b ¬ b ‹b ¬ b ‹Y’› b› b › b› b},

where Y’ is identical to Y except that the first entry has been reduced by 1.

For the case [X] = [1 ¬ 1 \ 2], the θ(Ω^θ(Ω^θ(1))) level separator is defined as follows:

{a, b [1 [1 ¬ 1 [1 ¬ 1 \ 2] 2] 2] 2} = {a ‹0 [1 ¬ 1 [1 ¬ 1 \ 2] 2] 2› b}

= {a ‹b ‹ b ¬ b ‹b ¬ R_b› b › b› b},

where R_n = ‘b ‹R_{n-1}› b’,

R_1 = ‘0’.

The complete Angle Bracket (A) Rules can now be rewritten to incorporate the modified additional Angle Bracket (B) Rules (now in Rules A2 and A5):-

Rule A1 (only 1 entry of either 0 or 1):

‘a ‹0› b’ = ‘a’,

‘a ‹1› b’ = ‘a, a, ... , a’ (with b a’s).

Rule A2 (only 1 entry of either 0 or 1 prior to ¬ symbol):

‘a ‹0 ¬ N› b’ = ‘a’,

‘a ‹1 ¬ N› b’ = ‘a [1¬N] a [1¬N] ... [1¬N] a’ (with b a’s),

‘a ‹1 ¬ 2› b’ = ‘a \ a \ ... \ a’ (with b a’s).

Rule A3 (last entry in any 1-space or higher dimensional space of array is 1):

‘a ‹# [A] 1› b’ = ‘a ‹#› b’ (# can include the ¬ symbol in this equation),

‘a ‹# [A] 1 ¬ N› b’ = ‘a ‹# ¬ N› b’,

‘a ‹# ¬ 1› b’ = ‘a ‹#› b’.

When [A] is a normal separator and [B] is a hyperseparator, or level of [A] is less than level of [B],

‘a ‹# [A] 1 [B] #*› b’ = ‘a ‹# [B] #*› b’ (either # or #* can include the ¬ symbol here).

Remove trailing 1’s.

Rule A4 (number to right of angle brackets is 1):

‘a ‹A› 1’ = ‘a’.

Rule A5 (Rules A1-4 do not apply, first entry is 0, separator immediately prior to next non-1 entry (c_1) is [A_{1,p_1}]):

‘a ‹ 0 [$A_{1,1}$] 1 [$A_{1,2}$] ... 1 [A_{1,p_1}] c_1 $\#_1$ $\#_0$ › b’ = ‘a ‹S_1 $\#_0$› b’,

where $p_1 \geq 1$ and $\#_0$ is either ‘¬ N’ or an empty string (case N = ‘1’).
Set i = 1 and follow Rules A5a-d (b, c and d are terminal, a is not).

Rule A5a (separator [A_{i,p_i}] = [1 ¬ 1 [$A_{i+1,1}$] 1 [$A_{i+1,2}$] ... 1 [$A_{i+1,p_{i+1}}$] c_{i+1} $\#_{i+1}$], where $p_{i+1} \geq 1$):

S_i = ‘b ‹$A_{i,1}$’› b [$A_{i,1}$] b ‹$A_{i,2}$’› b [$A_{i,2}$] ... b ‹A_{i,p_i-1}’› b [A_{i,p_i-1}] b ‹b ¬ S_{i+1}› b [A_{i,p_i}] c_i-1 $\#_i$’.

Increment i by 1 and repeat Rules A5a-d.

Rule A5b (separator [A_{i,p_i}] = [1 ¬ 2] = \):

S_i = ‘R_b’,

R_n = ‘b ‹$A_{i,1}$’› b [$A_{i,1}$] b ‹$A_{i,2}$’› b [$A_{i,2}$] ... b ‹A_{i,p_i-1}’› b [A_{i,p_i-1}] b ‹R_{n-1}› b \ c_i-1 $\#_i$’ (n > 1),

R_1 = ‘0’.

Rule A5c (Rules A5a-b do not apply, separator [A_{i,p_i}] = [1 ¬ k #*], where k ≥ 2):

S_i = ‘b ‹$A_{i,1}$’› b [$A_{i,1}$] b ‹$A_{i,2}$’› b [$A_{i,2}$] ... b ‹A_{i,p_i-1}’› b [A_{i,p_i-1}] b ‹R_b› b [1 ¬ k #*] c_i-1 $\#_i$’,

R_n = ‘b ‹$A_{i,1}$’› b [$A_{i,1}$] b ‹$A_{i,2}$’› b [$A_{i,2}$] ... b ‹A_{i,p_i-1}’› b [A_{i,p_i-1}] b ‹R_{n-1}› b [1 ¬ k #*] c_i-1 $\#_i$ ¬ k-1 #*’ (n > 1),

R_1 = ‘0’.

Rule A5d (Rules A5a-c do not apply):

S_i = ‘b ‹$A_{i,1}$’› b [$A_{i,1}$] b ‹$A_{i,2}$’› b [$A_{i,2}$] ... b ‹A_{i,p_i}’› b [A_{i,p_i}] c_i-1 $\#_i$’.

Rule A6 (Rules A1-5 do not apply):

‘a ‹n #› b’ = ‘a ‹n-1 #› b [n #] a ‹n-1 #› b [n #] ... [n #] a ‹n-1 #› b’

(with b ‘a ‹n-1 #› b’ strings, # can include the ¬ symbol).

Notes:

1. A, B, $A_{i,1}$, $A_{i,2}$, ... , A_{i,p_i} are strings of characters within separators.
2. $A_{i,1}$’, $A_{i,2}$’, ... , A_{i,p_i}’ are strings of characters within angle brackets that are identical to the strings $A_{i,1}$, $A_{i,2}$, ... , A_{i,p_i} respectively except that the first entries of each have been reduced by 1. If $A_{i,j}$ (for some $1 \leq j \leq p_i$) begins with 1, $A_{i,j}$’ begins with 0.
3. N is a string of characters that follow the ¬ symbol.
4. S_i and R_n are string building functions which create strings of characters. The latter function nests the same string of characters around itself n-1 times before being replaced by the string ‘0’.
5. #, #* and $\#_i$ are strings of characters representing the remainder of the array (can be null or empty), which excludes the ¬ symbol (unless otherwise stated).
6. The backslash (\) is a hyperseparator. The ¬ symbol enclosed by one pair of square brackets (e.g. [A¬B]) is also a hyperseparator. The ¬ symbol itself is a 2-hyperseparator. All other separators are normal separators.
7. The comma is used as shorthand for the [1] separator.
8. The backslash (\) is used as shorthand for the [1¬2] separator.

This is the third stage of my Hyper-Nested Array Notation. The limit ordinal of this notation is θ(Ω^Ω), the large Veblen ordinal.

In Rule A3, the levels of two hyperseparators [A] = [A_1¬A_2] and [B] = [B_1¬B_2] are determined by the ordinal levels of the two normal separators [A_2] and [B_2] – if [A_2] ranks lower than [B_2], then [A] ranks lower than [B] and the '[A] 1' string is deleted. If [A_2] and [B_2] are identical, then the levels of [A] and [B] are determined by the ordinal levels of the two normal separators [A_1] and [B_1] – if the level of [A_1] is less than that of [B_1], then [A] ranks lower than [B].

In Rule A5, we first consider the [A_{1,p_1}] separator, which is immediately prior to the next non-1 entry (after the initial 0) in the angle brackets of the left hand side of the initial 'replacement equation' (after performing that equation). If it is a nesting separator, Rule A5b or A5c is executed; if it is non-special, Rule A5d is employed. If, however, [A_{1,p_1}] is a branching separator, Rule A5a is performed, and we repeat A5a-d for the [A_{2,p_2}] separator enclosed within [A_{1,p_1}]. If [A_{2,p_2}] is also branching, A5a is again performed, and we repeat A5a-d for [A_{3,p_3}] within [A_{2,p_2}], and so on, until we find a [A_{i,p_i}] (for some i) that is non-branching.

The Main (M) Rules (when no angle brackets ‹ › appear in the main array) remain the same as in my Nested Array Notation, as they only process the 'base layer' of the main array (within curly brackets); the [A_i] separators in the 'base layer', for example,

{n_1 [A_1] n_2 [A_2] ... [A_k] n_{k+1}},

are always normal separators, although the A_i strings can include the special symbols that go beyond my Nested Array Notation (e.g. \ and ¬). The Angle Bracket (A) Rules, on the other hand, process the layers above the 'base layer' of the main array, which are the 'base layers' of angle bracket arrays (within angle brackets), and have to adapt to more advanced notation such as hyperseparators.

The next step would be to define a separator with more than one ¬ symbol,

{a, b [1 [1 ¬ 1 ¬ 2] 2] 2} = {a ‹0 [1 ¬ 1 ¬ 2] 2› b}
= {a ‹b ‹b ¬ b ‹b ¬ b ‹ ... ‹b ¬ b ‹b ¬ b› b› ... › b› b› b› b}
(with b angle brackets and b-1 ¬ symbols)
= {a ‹b ‹R_b› b› b},

where R_n = 'b ¬ b ‹R_{n-1}› b',
R_1 = '0'.

This is equivalent to

{a, b [1 [1 ¬ 1 [1 ¬ 1 [... [1 ¬ 1 [1¬1,2] 2] ...] 2] 2] 2] 2} (with b-1 ¬ symbols, for b ≥ 2)

and is a θ(Ω^Ω)-recursive function.

Notice the similarities with the lowest separator with more than one backslash (introduced in page 11 of Beyond Bird's Nested Arrays I),

{a, b [1 \ 1 \ 2] 2} = {a ‹0 \ 1 \ 2› b}
= {a ‹b \ b ‹b \ b ‹ ... ‹b \ b ‹b \ b› b› ... › b› b› b}
(with b-1 angle brackets and b-1 \ symbols)
= {a ‹R_b› b},

where R_n = 'b \ b ‹R_{n-1}› b',
R_1 = '0'.

That is equivalent to

{a, b [1 \ 1 [1 \ 1 [... [1 \ 1 [1 \ 1, 2] 2] ...] 2] 2] 2} (with b-1 \ symbols, for b ≥ 2).

[1 [1 ¬ 1 ¬ 2] 2] has level θ(Ω^Ω).

For example,

{3, 2 [1 [1 ¬ 1 ¬ 2] 2] 2} = {3 ‹0 [1 ¬ 1 ¬ 2] 2› 2}

$= \{3 \langle 2 \langle 2 \neg 2 \rangle 2 \rangle 2\}$
$= \{3 \langle 2 \backslash 2 [2 \neg 2] 2 \backslash 2 \rangle 2\}$,
$\{3, 3 [1 [1 \neg 1 \neg 2] 2] 2\} = \{3 \langle 0 [1 \neg 1 \neg 2] 2 \rangle 3\}$
$= \{3 \langle 3 \langle 3 \neg 3 \langle 3 \neg 3 \rangle 3 \rangle 3 \rangle 3\}$,
$\{3, 4 [1 [1 \neg 1 \neg 2] 2] 2\} = \{3 \langle 0 [1 \neg 1 \neg 2] 2 \rangle 4\}$
$= \{3 \langle 4 \langle 4 \neg 4 \langle 4 \neg 4 \langle 4 \neg 4 \rangle 4 \rangle 4 \rangle 4 \rangle 4\}$.

But this is a whole new hyperlevel of gigantic numbers, better left for the next part, in which I aim to reach even higher ordinals.

Amazingly, these ordinals can be expressed using just ω and my various array notations. Generally, the ordinal $\theta(\Omega^\wedge\alpha)$ for some ordinal α would require an array notation with recursion level α (or α arguments). As already shown on page 3 of Beyond Bird's Nested Arrays I,

$\varphi(\alpha_n, \dots , \alpha_3, \alpha_2, \alpha_1) = \{\omega, \omega(1+\alpha_1), 1+\alpha_2, 1+\alpha_3, \dots , 1+\alpha_n\}$ (for ordinals not less than ε_0),
$\theta(\Omega^\wedge n) = \varphi(1, 0, 0, \dots , 0)$ (with n+1 zeroes)
$= \{\omega, \omega, 1, \dots , 1, 2\}$ (with n 1's)
$> \{\omega, \omega, \omega, \dots , \omega\}$ (with n+2 ω's),
$\theta(\Omega^\wedge\omega) = \{\omega, \omega [2] 2\} = \{\omega \langle 1 \rangle \omega\}$
$= \{\omega, \omega, \omega, \dots , \omega\}$ (with ω omegas).

Since $\theta(\Omega^\wedge\omega)$ represents the limit of the size of a linear array (maximum ω entries), I need to use multi-dimensional arrays in order to go beyond this ordinal. We get the following sequence:

$\varepsilon_{\theta(\Omega^\wedge\omega)+1} = \omega^{\wedge\wedge}(\omega(\theta(\Omega^\wedge\omega)+1))$
$= \{\omega, \theta(\Omega^\wedge\omega)+\omega, 2\}$,
$\Gamma_{\theta(\Omega^\wedge\omega)+1} = \{\omega, \theta(\Omega^\wedge\omega)+\omega, 1, 2\}$,
$\theta(\Omega^\wedge 2, \theta(\Omega^\wedge\omega)+1) = \{\omega, \theta(\Omega^\wedge\omega)+\omega, 1, 1, 2\}$,

etc., where the (n+1)th term is

$\theta(\Omega^\wedge n, \theta(\Omega^\wedge\omega)+1) = \{\omega, \theta(\Omega^\wedge\omega)+\omega, 1, \dots , 1, 1, 2\}$ (with n 1's).

Something is amiss, since we would only be 'feeding back' higher and higher ordinals into the second entries of the linear arrays, without increasing their lengths. Since

$\{\omega, \omega+1 [2] 2\} = \{\omega, \omega, \omega, \dots , \omega\}$ (with a maximum of ω omegas)
$= \{\omega, \omega [2] 2\} = \theta(\Omega^\wedge\omega)$,

I have redefined the above so that

$\{\omega, \omega+1 [2] 2\} = \varepsilon_{\theta(\Omega^\wedge\omega)+1}$,

and, for $1 \le n < \omega$,

$\{\omega, \omega+n [2] 2\} = \theta(\Omega^\wedge(n-1), \theta(\Omega^\wedge\omega)+1)$,

and that $f(\alpha) = \{\omega, \alpha [2] 2\}$ is a strictly increasing function for transfinite α. The growth of $\{\omega, \omega+n [2] 2\}$ for finite n is analogous to that of $\{\omega, n [2] 2\}$ since, for $2 \le n < \omega$,

$\{\omega, n [2] 2\} < \theta(\Omega^\wedge(n-2), 0) = \theta(\Omega^\wedge(n-2))$,

and the main difference is in the second argument of the 2-argument θ function (the starting value).

The $\{\omega, \omega+1 [2] 2\}$ problem is not unlike the $\omega^{\wedge\wedge}(\omega+1) = \{\omega, \omega+1, 2\}$ problem on the first page of Beyond Bird's Nested Arrays I. For $1 \le n < \omega$,

$\{\omega, n, 2\} = \omega^\wedge\omega^\wedge\omega^\wedge\dots^\wedge\omega^\wedge 1$ (with n omegas, starting value 1),
$\{\omega, \omega+n, 2\} = \omega^\wedge\omega^\wedge\omega^\wedge\dots^\wedge\omega^\wedge(\varepsilon_0+1)$ (with n omegas, starting value ε_0+1).

In general,

$\{\omega, \omega(1+\alpha)+n, 2\} = \omega^\wedge\omega^\wedge\omega^\wedge\dots^\wedge\omega^\wedge(\varepsilon_\alpha+1)$ (with n omegas, starting value $\varepsilon_\alpha+1$).

Since $\theta(\Omega^n, \theta(\Omega^\omega, \alpha)+1)$ has limit ordinal $\theta(\Omega^\omega, \alpha+1)$ as n tends to ω, it follows that

$\theta(\Omega^\omega, 1) = \{\omega, \omega 2 [2] 2\}$,
$\theta(\Omega^n, \theta(\Omega^\omega, 1)+1) = \{\omega, \omega 2+n+1 [2] 2\}$,
$\theta(\Omega^\omega, \alpha) = \{\omega, \omega(1+\alpha) [2] 2\}$,
$\theta(\Omega^n, \theta(\Omega^\omega, \alpha)+1) = \{\omega, \omega(1+\alpha)+n+1 [2] 2\}$,
$\theta(\Omega^\omega, \theta(\Omega^\omega)) = \{\omega, \{\omega, \omega [2] 2\} [2] 2\}$
$= \{\omega, \{\omega, 2, 2 [2] 2\} [2] 2\} = \{\omega, 3, 2 [2] 2\}$,
$\theta(\Omega^\omega, \theta(\Omega^\omega, \theta(\Omega^\omega))) = \{\omega, 4, 2 [2] 2\}$,
$\theta(\Omega^\omega+1) = \{\omega, \omega, 2 [2] 2\}$,

and we would get

$\theta(\Omega^\omega+\Omega^n) = \{\omega, \omega, 1, \ldots , 1, 2 [2] 2\}$ (with n 1's)
$> \{\omega, \omega, \omega, \ldots , \omega [2] 2\}$ (with n+2 ω's),
$\theta((\Omega^\omega)2) = \{\omega, \omega [2] 3\} = \{\omega \langle 1 \rangle \omega [2] 2\}$,
$\theta((\Omega^\omega)\alpha) = \{\omega, \omega [2] 1+\alpha\} = \{\omega \langle 1 \rangle \omega [2] \alpha\}$,
$\theta((\Omega^\omega)\theta(\Omega^\omega)) = \{\omega \langle 1 \rangle \omega [2] \{\omega \langle 1 \rangle \omega\}\}$
$> \{\omega, \omega [2] \omega\} = \{\omega, 2 [2] 1, 2\}$,
$\theta((\Omega^\omega)\theta((\Omega^\omega)\theta(\Omega^\omega))) > \{\omega \langle 1 \rangle \omega [2] \{\omega, 2 [2] 1, 2\}\} > \{\omega, 3 [2] 1, 2\}$,
$\theta(\Omega^{\omega+1}) = \{\omega, \omega [2] 1, 2\}$,
$\theta(\Omega^{\omega+1}, \alpha) = \{\omega, \omega(1+\alpha) [2] 1, 2\}$,
$\theta(\Omega^{\omega+1}+1) = \{\omega, \omega, 2 [2] 1, 2\}$,
$\theta(\Omega^{\omega+1}+\Omega^\omega) = \{\omega, \omega [2] 2, 2\} = \{\omega \langle 1 \rangle \omega [2] 1, 2\}$,
$\theta((\Omega^{\omega+1})2) = \{\omega, \omega [2] 1, 3\}$,
$\theta((\Omega^{\omega+1})\alpha) = \{\omega, \omega [2] 1, 1+\alpha\}$,
$\theta(\Omega^{\omega+2}) = \{\omega, \omega [2] 1, 1, 2\}$,
$\theta(\Omega^{\omega+n}) = \{\omega, \omega [2] 1, \ldots , 1, 2\}$ (with n 1's)
$> \{\omega \langle 1 \rangle \omega [2] \omega \langle 1 \rangle n\}$,
$\theta(\Omega^{\omega 2}) = \{\omega \langle 1 \rangle \omega [2] \omega \langle 1 \rangle \omega\}$.

Since no α-dimensional space in an array may exceed ω^α entries, we would obtain

$\theta(\Omega^{\omega 3}) = \{\omega \langle 1 \rangle \omega [2] \omega \langle 1 \rangle \omega [2] \omega \langle 1 \rangle \omega\}$,
$\theta(\Omega^{\omega^2}) = \{\omega, \omega [3] 2\} = \{\omega \langle 2 \rangle \omega\}$,
$\theta(\Omega^{\omega^3}) = \{\omega, \omega [4] 2\} = \{\omega \langle 3 \rangle \omega\}$,
$\theta(\Omega^{\omega^\omega}) = \{\omega, \omega [1, 2] 2\} = \{\omega \langle \omega \rangle \omega\}$,
$\theta(\Omega^{\omega^{\omega^2}}) = \{\omega, \omega [1, 1, 2] 2\} = \{\omega \langle \omega, \omega \rangle \omega\}$,
$\theta(\Omega^{\omega^{\omega^\omega}}) = \{\omega, \omega [1 [2] 2] 2\} = \{\omega \langle \omega \langle 1 \rangle \omega \rangle \omega\}$,
$\theta(\Omega^{\omega^{\omega^{\omega^\omega}}}) = \{\omega, \omega [1 [1, 2] 2] 2\} = \{\omega \langle \omega \langle \omega \rangle \omega \rangle \omega\}$,
$\theta(\Omega^{\omega^{\omega^{\omega^{\omega^\omega}}}}) = \{\omega, \omega [1 [1 [2] 2] 2] 2\} = \{\omega \langle \omega \langle \omega \langle 1 \rangle \omega \rangle \omega \rangle \omega\}$,
$\theta(\Omega^{\omega^{\omega^{\omega^{\omega^{\omega^\omega}}}}}) = \{\omega, \omega [1 [1 [1, 2] 2] 2] 2\} = \{\omega \langle \omega \langle \omega \langle \omega \rangle \omega \rangle \omega \rangle \omega\}$.

For $\alpha \geq \varepsilon_0$ (i.e. $\alpha \geq \theta(1)$),

$\theta(\Omega^\alpha) = \{\omega, \omega [X] 2\}$,

where the separator [X] has level α. Since the separator with level $\theta(\Omega^\alpha)$ is $[1 [1 \neg X] 2]$, it follows that the transfinite ordinal array notation 'catches up' with its regular use for finite numbers at the large Veblen ordinal.

For example,

$\theta(\Omega^{\theta(1)}) = \{\omega, \omega [1 \backslash 2] 2\}$
$= \{\omega \langle \omega \langle \omega \langle \ldots \langle \omega \langle \omega \rangle \omega \rangle \ldots \rangle \omega \rangle \omega \rangle \omega\}$ (with ω ω's from centre to right),
$\theta(\Omega^{\theta(2)}) = \{\omega, \omega [1 \backslash 1 \backslash 2] 2\}$,

θ(Ω^θ(ω)) = {ω, ω [1 [2 ¬ 2] 2] 2},
θ(Ω^θ(Ω)) = {ω, ω [1 [1 ¬ 3] 2] 2},
θ(Ω^θ(Ω^ω)) = {ω, ω [1 [1 ¬ 1, 2] 2] 2},
θ(Ω^θ(Ω^θ(1))) = {ω, ω [1 [1 ¬ 1 \ 2] 2] 2},
θ(Ω^θ(Ω^θ(Ω))) = {ω, ω [1 [1 ¬ 1 [1 ¬ 3] 2] 2] 2},
θ(Ω^θ(Ω^θ(Ω^ω))) = {ω, ω [1 [1 ¬ 1 [1 ¬ 1, 2] 2] 2] 2},
θ(Ω^θ(Ω^θ(Ω^θ(1)))) = {ω, ω [1 [1 ¬ 1 [1 ¬ 1 \ 2] 2] 2] 2},
θ(Ω^θ(Ω^θ(Ω^θ(Ω^θ(1))))) = {ω, ω [1 [1 ¬ 1 [1 ¬ 1 [1 ¬ 1 \ 2] 2] 2] 2] 2},
θ(Ω^θ(Ω^θ(Ω^θ(Ω^θ(Ω^θ(1)))))) = {ω, ω [1 [1 ¬ 1 [1 ¬ 1 [1 ¬ 1 [1 ¬ 1 \ 2] 2] 2] 2] 2] 2},

and at the limit ordinal,

θ(Ω^Ω) = {ω, ω [1 [1 ¬ 1 ¬ 2] 2] 2}.

Jonathan Bowers has made an attempt at creating an ‘Extended Array Notation’ that goes beyond linear arrays on his website but the rules have not been clearly defined. Instead he has made references to ‘Tetrational Arrays’ and ‘Pentational Arrays’ that might be comparable to my Nested Arrays and slightly beyond them (e.g. with separators of the form [X \ Y]) respectively. A number he describes as a ‘Triakulus’ on his ‘Infinity Scrapers’ page (www.polytope.net/hedrondude/scrapers.htm) is defined as a 3^^^3 array of 3’s – it is difficult to define this using my hierarchy of array notations but I guess that it might be the number

{3, 3 [1 \ 1 \ 2] 2} = {3 ‹0 \ 1 \ 2› 3}
= {3 ‹3 \ 3 ‹3 \ 3› 3› 3}.

Bowers has introduced ‘Legion Arrays’ (very vaguely defined) on his ‘Exploding Array Function’ page (www.polytope.net/hedrondude/array.htm). These begin with small linear arrays followed by a forward slash and 2, as in

{b, p / 2} = (...(((b array of b’s) array of b’s) array of b’s) ...) array of b’s (with p b’s).

For example,

{3, 2 / 2} = 3 array of 3’s = {3, 3, 3} = 3^^^3,
{3, 3 / 2} = 3^^^3 array of 3’s (his ‘Triakulus’).

His forward slash might have been the equivalent of my [1 [1 ¬ 1 ¬ 2] 2] separator if it was properly defined.

Back to my Hyper-Nested Array Notation (beginning of stage 4). Has anyone ever heard of a function whose growth rate is of the magnitude of the large Veblen ordinal (if not faster)?

The number

N = {3, 3 [1 [1 ¬ 1 ¬ 2] 2] 2}

is large enough, but

{3, 3, 2 [1 [1 ¬ 1 ¬ 2] 2] 2} = {3, {3,2,2 [1 [1¬1¬2] 2] 2} [1 [1¬1¬2] 2] 2}
= {3, N [1 [1 ¬ 1 ¬ 2] 2] 2}
= {3 ‹0 [1 ¬ 1 ¬ 2] 2› N}
= {3 ‹N ‹N ¬ N ‹N ¬ N ‹ ... ‹N ¬ N ‹N¬N› N› ... › N› N› N› N},

in which there are N angle brackets and N-1 ¬ symbols. Truly awesome!

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Last modified: 6 September 2017
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